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Interaction between compacton and anticompacton, peakon and antipeakon in $(2+1)$-dimensional spaces*

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Starting from the variable separation solution obtained by using the extended homogenous balance method, a class of novel localized coherent structures such as the multi-peakon–antipeakons solution and the multi-compacton–anticompactons solution of the $(2+1)$-dimensional dispersive long wave equation are found by selecting appropriate functions. These new structures exhibit some novel interaction features that are different from one of the known results. Their interaction behaviour is very similar to the completely elastic collisions between two classical particles.

Keywords: homogeneous balance method, compacton, peakon, $(2+1)$-dimensional dispersive long wave equation
PACC: 0230, 0340, 0545

1. Introduction

The integrable dispersive long wave equations are interesting topics in physics and mathematics, especially in fluid mechanics. In recent four decades, many a (1+1)-dimensional versions of the dispersive long wave equations have been proposed to model the water wave propagation in certain infinitely long channels of finite constant depth and narrow width.[1–4] Recently, to cover wide channels or open seas, that system has been extended to coupled integrable dispersive long wave equations (IDLWEs) in $(2+1)$-dimensional spaces, as a compatibility condition for a weak Lax pair,[5]

$$u_{ty} = -\eta_{xx} - \frac{1}{2}(u^2)_{xy}, \tag{1}$$

$$\eta_t = (u\eta + u + u_{xy})_x, \tag{2}$$

where $\eta(x,y,t)$ represents the amplitude of a surface wave, which propagates in the (x,y) plane, with its horizontal velocity $u(x,y,t)$. A relevant Kac–Moody–Virasoro symmetry algebra has been studied.[6] Nine type of two-dimensional partial differential equation reductions and 13 types of ordinary differential equations by means of the direct and non-classical method are obtained.[7] A set of generalized symmetries with arbitrary functions of time[8] by a simple constructive formula constitutes an infinite-dimensional Lie algebra. It is interesting to note that the IDLWEs have no Painlevé property even if they are integrable.[9] A kind of special multisoliton-like solutions[10,11] and abundant localized coherent structures[12] for the IDLWE are obtained by the extended homogeneous balance method.[13]

It is well known that, in addition to the continuous localized excitations in (1+1)-dimensional nonlinear systems, some types of significant weak solutions like the peakon and compacton have attracted much attention of both mathematicians and physicists. The so-called peakon solution ($u = c\exp(-|x-ct|)$) which is called a weak solution of the celebrated (1+1)-dimensional Camassa–Holm equation

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \tag{3}$$

was first given by Camassa and Holm.[14] While the so-called (1+1)-dimensional compacton solution was

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firstly given by Rosenau and Hyman.[15] It can describe the typical (1+1)-dimensional soliton solutions with a finite wavelength when the nonlinear dispersion effects are included. It has been found that peakon and compacton may have many interesting properties and possible physical applications.[16] Recently, the higher-dimensional peakon solution and compacton solution have also been investigated and obtained in some $(2+1)$-dimensional models.[17−20] In this paper, taking the $(2+1)$-dimensional IDLWE as example, we are interested in the study of the explicit formulae of multi-compactons-anticompactons and multi-peakons and antipeakons solutions; at the same time we make a detailed analysis of the interactions between a single compacton and single anticompacton, a double compactons and a double anticompactons, a single peakon and single antipeakon, a double peakons and double antipeakons respectively.

This paper is organized as follows. In Section 2 of this paper, a brief review on the variable separation solution for the $(2+1)$-dimensional IDLWE is described. Section 3 and Section 4 are devoted to find the multi-compactons-anticompactons, and multi-peakon–antipeakon solutions and discuss their interactions. The final section is a short summary.

2. Variable separation solutions for the $(2+1)$-dimensional IDLWE

For completeness, we make a short review of the variable separation procedure here by using the extended homogeneous balance method.[21]

By means of the leading order analysis, we can obtain the following Bäcklund transformations

$$\begin{aligned} u &= f'(\omega)\omega_x + u_1(x,y,t), \\ \eta &= g''(\omega)\omega_x\omega_y + g'(\omega)\omega_{xy} + \eta_1(x,y,t), \end{aligned} \tag{4}$$

where $f(\omega)$, $g(\omega)$ and $\omega\ (x,\ y,\ t)$ are suitable functions of $\{\omega\}$ and $\{x,\ y,\ t\}$ respectively, $u_1\ (x,\ y,\ t)$ and $\eta_1(x,\ y,\ t)$ are known solutions of the $(2+1)$-dimensional IDLWE. Substituting the Bäcklund transformation (4) into the $(2+1)$-dimensional IDLWE, we can simplify them into the form as follows:

$$\omega_t \pm \omega_{xx} + u_1\omega_x = 0, \tag{5}$$

$$\eta_1 \mp u_{1y} + 1 = 0, \tag{6}$$

where $f(\omega)$ and $g(\omega)$ satisfy

$$f(\omega) = \pm 2\ln\omega, \quad g(\omega) = 2\ln\omega. \tag{7}$$

By analysing the $(2+1)$-dimensional IDLWE, we may find a special known solution of the $(2+1)$-dimensional IDLWE

$$u_1 = h(x,t), \quad \eta_1 = 1. \tag{8}$$

So from the above equation, we have

$$\omega_t + \omega_{xx} \pm 2h(x,t)\omega_x = 0. \tag{9}$$

Because the seed solution $h(x,t)$ in Eq.(9) is an arbitrary function of variables (x,t), we can find the general variable separation solutions as follows:

$$\begin{aligned} &\omega(x,y,t) = a_0 + a_1 p(x,t) + a_2 q(y) + a_3 p(x,t)q(y), \\ &h(x,t) = -\frac{p_t \pm p_{xx}}{2p_x}, \end{aligned} \tag{10}$$

where $p \equiv p(x,t)$ and $q \equiv q(y)$ are arbitrary functions of $\{x,t\}$ and $\{y\}$ respectively, a_0, a_1, a_2, a_3 are arbitrary constants. Substituting solution (7) with Eq.(10) into Eqs.(4) and (5), we can obtain the variable separation solution of the $(2+1)$-dimensional IDLWE[11]

$$u(x,y,t) = \pm\frac{(a_1 + a_3 q)p_x}{a_0 + a_1 p + a_2 q + a_3 pq} - \frac{p_t \pm p_{xx}}{2p_x}, \tag{11}$$

$$v = \eta(x,y,t) + 1 = \frac{(a_0 a_3 - a_1 a_2)p_x q_y}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \tag{12}$$

3. The combined multi-compacton and multi-anticompacton structures and their interaction behaviour

Because of the entrance of arbitrary functions in solution (12), we can find some types of multi-compacton solution for the (2+1)-dimensional IDLWE by selecting the arbitrary functions appropriately. For instance, if we fixed the functions p and q as

$$p = \sum_{i=1}^{M} \begin{cases} 0, & x + v_i t \le x_{0i} - \dfrac{\pi}{2k_i}, \\ b_i \sin(k_i(x + v_i t - x_{0i})) + b_i, & x_{0i} - \dfrac{\pi}{2k_i} < x + v_i t \le x_{0i} + \dfrac{\pi}{2k_i}, \\ 2b_i, & x + v_i t > x_{0i} + \dfrac{\pi}{2k_i}, \end{cases} \tag{13}$$

and

$$q=\sum_{j=1}^{N}\begin{cases}0, & y\le y_{0j}-\dfrac{\pi}{2l_i},\\ c_j\sin(l_j(y-y_{0j}))+c_j, & y_{0j}-\dfrac{\pi}{2l_j}<y\le y_{0j}+\dfrac{\pi}{2l_j},\\ 2c_i, & y>y_{0j}+\dfrac{\pi}{2l_j},\end{cases}\tag{14}$$

in expressions (13) and (14),where $b_i, k_i, v_i, c_j, l_j, x_{0i}$ and y_{0j} are all arbitrary constants, then the solution (12) with Eqs.(13) and (14) becomes a multi-compactons-anticompactons solution.

From Eqs.(13) and (14), we can see that the piecewise functions p and q of the compacton solutions are once differential

$$p_x=\sum_{i=1}^{M}\begin{cases}0, & x+v_it\le x_{0i}-\dfrac{\pi}{2k_i},\\ b_ik_i\cos(k_i(x+v_it-x_{0i}))+b_i, & x_{0i}-\dfrac{\pi}{2k_i}<x+v_it\le x_{0i}+\dfrac{\pi}{2k_i},\\ 0, & x+v_it>x_{0i}+\dfrac{\pi}{2k_i},\end{cases}\tag{15}$$

and

$$q_y=\sum_{j=1}^{N}\begin{cases}0, & y\le y_{0j}-\dfrac{\pi}{2l_j},\\ c_jl_j\cos(l_j(y+y_{0j}))+c_j, & y_{0j}-\dfrac{\pi}{2l_j}<y\le y_{0j}+\dfrac{\pi}{2l_j}.\\ 0, & y>y_{0j}+\dfrac{\pi}{2l_j}.\end{cases}\tag{16}$$

When choosing $M=1$, $N=1$, $a_0=20$, $a_1=1$, $a_2=-1$, $a_3=2$, $b_1=-1$, $k_1=1$, $v_1=1$, $c_1=1$, $l_1=1$, $x_{01}=0$, $y_{01}=0$ and $t=3$, we can obtain a combined single compacton and single anticompacton solution for the $(2+1)$-dimensional IDLWE. Figures 1(a)–(d) show the evolution of the interaction, between the compacton and anticompacton.

If selecting $M=2$, $N=2$, $a_0=20$, $a_1=1$, $a_2=-1$, $a_3=\dfrac{1}{25}$, $b_1=1.2$, $b_2=1$, $c_1=c_2=1$, $k_1=k_2=1$, $v_1=-1$, $v_2=3$, $l_1=l_2=1$, $x_{01}=0$, $x_{02}=0$, $y_{01}=0$, $y_{02}=5$, then we can obtain a combined double compactons and double anticompactons solution for the $(2+1)$-dimensional IDLWE. Figures 2(a)–(d) show the evolution behaviour of their interaction.

From Figs.1 and 2, we find that the interaction between(or among) compacton(s) and anticompacton(s) exhibits a new phenomenon. The compacton–anticompacton interaction is perfectly elastic. This is not consistent with the interaction between two compactons described in Ref.[18].

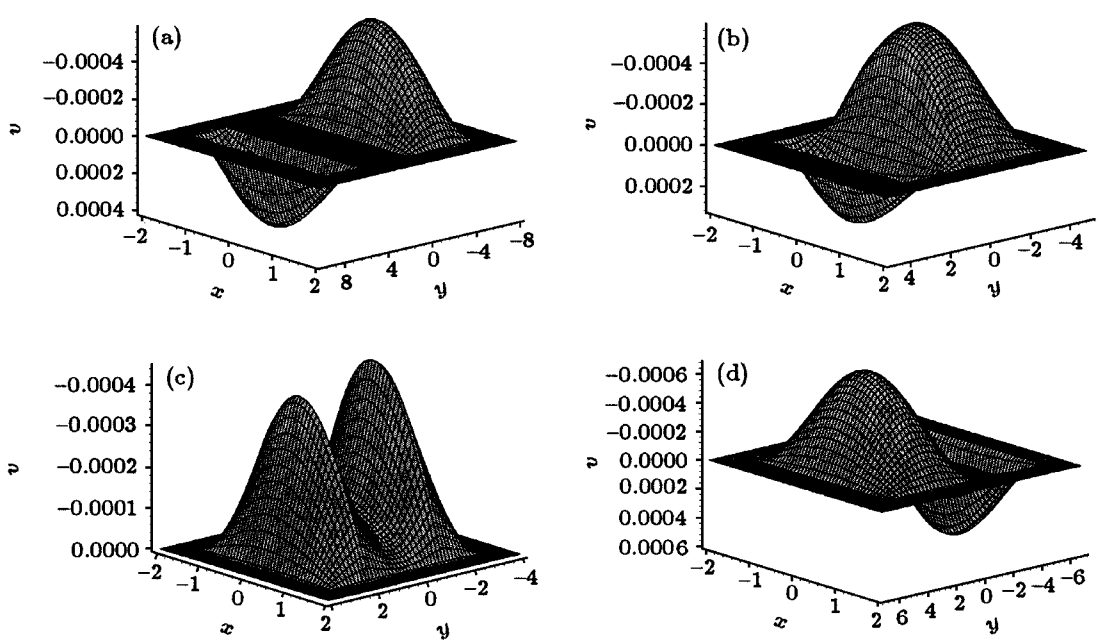


Fig.1. Evolution plot of a combined single-compacton and single anticompacton (a) $t=-3$, (b) $t=-1$, (c) $t=0$, (d) $t=2$.

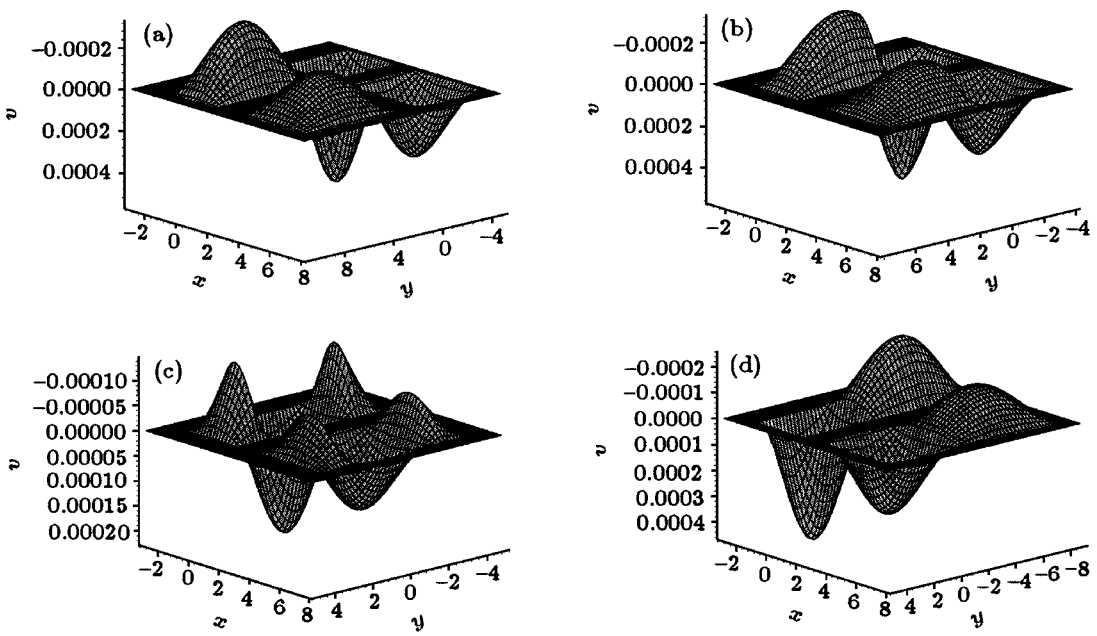


Fig.2. Evolution plot of a combined double compactons and anticompactons at (a) $t = -2$, (b) $t = -1$, (c) $t = 0$, (d) $t = 1$.

Similarly, considering the arbitrariness of the functions p and q in solution (12), we can construct a multi-peakon–antipeakon solution for the $(2+1)$-dimensional IDLWE by selecting appropriate functions. For instance, when p and q take the following simple form

$$p = \sum_{i=1}^{M} \begin{cases} d_i \exp(m_i x - \nu_i t + x_{0i}), & m_i x - \nu_i t + x_{0i} \le 0, \\ -d_i \exp(-m_i x + \nu_i t - x_{0i}), & m_i x - \nu_i t + x_{0i} > 0, \end{cases} \tag{17}$$

$$q = \sum_{j=1}^{N} \begin{cases} e_j \exp(n_j y + y_{0j}), & n_j y + y_{0j} \le 0, \\ -e_j \exp(-n_j y - y_{0j}) + 2y_{0j}, & n_j y + y_{0j} > 0, \end{cases} \tag{18}$$

where $d_i, m_i, \nu_i, e_j, n_j, x_{0i}$ and y_{0j} are all arbitrary constants, the solution (12) with Eqs.(17) and (18) becomes a combined multi-peakons and multi-antipeakons solution.

When choosing $M = 1$, $N = 3$, $a_0 = 5000$, $a_1 = 1$, $a_2 = 5$, $a_3 = 10$, $d_1 = 1$, $m_1 = 1$, $\nu_1 = 1$, $e_1 = e_2 = e_3 = 1$, $n_1 = n_2 = n_3 = 1$, $x_{01} = 4$, $y_{01} = y_{02} = 2$, $y_{03} = 3$ and $t = -10$, we can obtain a combined single peakon and single antipeakon solution for the $(2+1)$-dimensional IDLWE. Figures 3(a)–(d) show the evolution behaviour of their interaction.

If we select $M = 2$, $N = 2$, $a_0 = 5000$, $a_1 = a_2 = 1$, $a_3 = 2$, $d_1 = d_2 = 1$, $\nu_1 = \nu_2 = 1$, $m_1 = m_2 = 1$, $n_1 = n_2 = 1$, $e_1 = e_2 = 1$, $x_{01} = 4$, $x_{02} = -4$, $y_{01} = 4$, $y_{02} = -4$, we can obtain a combined double peakons and double antipeakons solution for the $(2+1)$-dimensional IDLWE. Figures 4(a)–(d) shows the evolution behaviour of their interaction.

From Figs.3 and 4, we also find that the interaction between(or among) peakon(s) and antipeakon(s) is completely elastic. This is also not consistent with the interaction between two peakons given in Ref.[17].

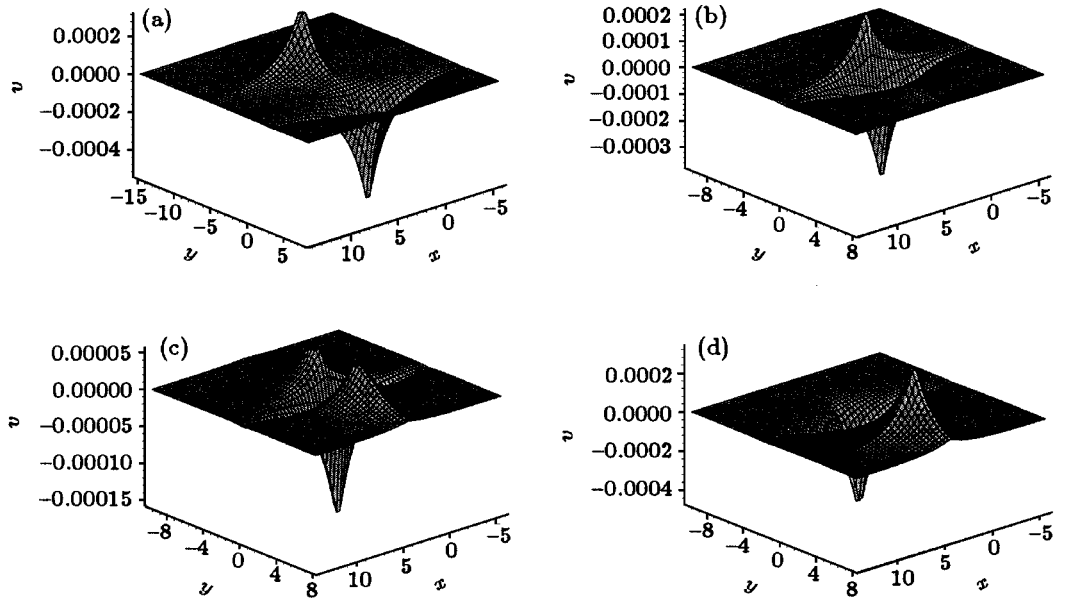


Fig.3. Evolution plot of a combined single peakon and single antipeakon at (a) $t = -5$, (b) $t = -2.5$, (c)$t = -2$, (d) $t = 0$.

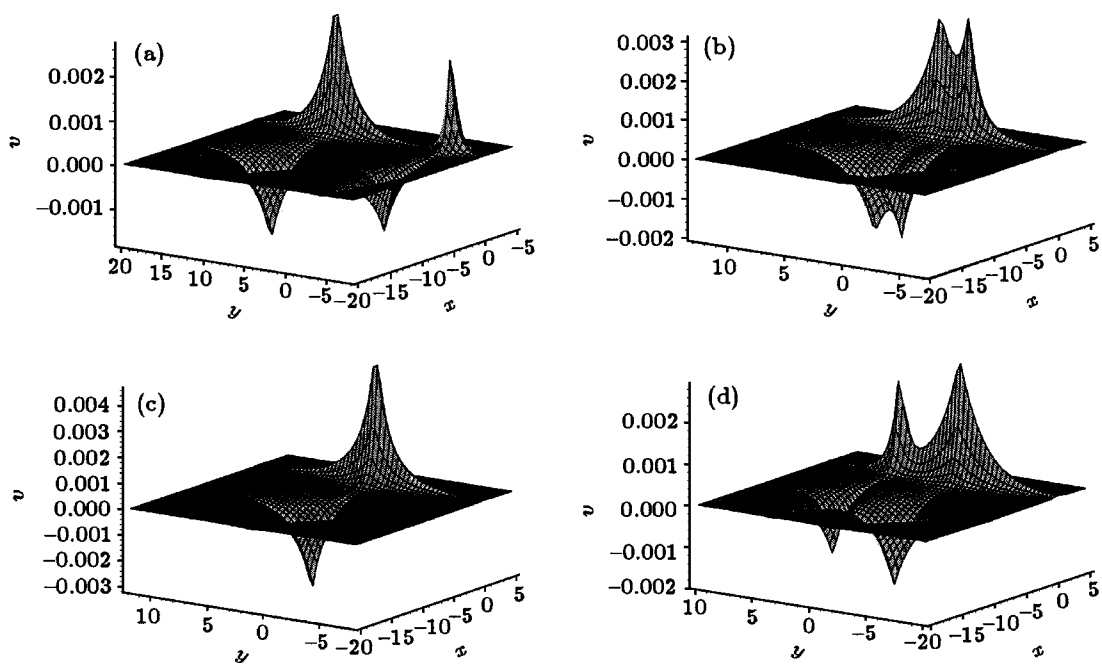


Fig.4. Evolution plot of a combined double peakons and double antipeakon solution at (a) $t = -5$, (b) $t = -1.5$, (c) $t = -1.15$, (d) $t = 0.5$.

4. Summary and discussion

In summary, starting from general variable separation solution for the $(2+1)$-dimensional IDLWE, we have constructed two kinds of new coherent structures (the combined peakons and antipeakons and the combined compactons and anticompactons) by selecting arbitrary functions appropriately. The interaction between (among) peakon(s) and antipeakon(s) or between (among) compacton(s) and anticompacton(s) exhibits interesting novel features which are not found in one-dimensional solitons. The compacton–anticompacton interaction and peakon–antipeakon interaction are all completely elastic. Their interaction behaviour is very similar to the completely elastic collisions between two classical particles.

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