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# Statistical properties of a single-mode laser driven by additive and multiplicative coloured noises with a coloured cross-correlation for different correlation times*

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We study a system for a single-mode laser driven by additive and multiplicative coloured noises with a coloured cross-correlation. The analytical expression of the stationary intensity distribution (SID) for the laser is derived in the case of three different correlation times. The influences of each stochastic parameter on the SID are discussed, the effects of the noise "colour" and the cross-correlation of noises on the mean intensity $< I >$, the variance, $\lambda_2(0)$, and the skewness, $\lambda_3(0)$ of the single-mode laser are investigated. We find that there are colourful phase transitions for the SID above a threshold, and re-entrant transitions induced by the "colour" of the additive noises. Further research of the curves of $< I >$-$\tau_i$, $\lambda_2(0)$-$\tau_i$, and $\lambda_3(0)$-$\tau_i$ ($i$=1,2,3) (where $\tau_i$ is the correlation time) shows that the output intensity not only increases with the additive noise correlation time $\tau_2$ and the cross-correlation time $\tau_3$, but also the quality of the output of laser beams is optimized.

**Keywords:** single-mode laser, coloured cross-correlation, statistical properties
**PACC:** 4250, 0540, 4260K

## 1. Introduction

Recently, the statistical properties of fluctuation in a single-mode laser have attracted considerable attention from both theorists and experimentalists.[1−6] In most of the existing theoretical studies, the multiplicative noise (pump noise) and the additive noise (quantum noise) are both considered as Gaussian white noises. In this approximate theoretical treatment, a lot of work has been done, and the results are generally in agreement with experimental findings.[7−9] Until 1991, Fulinski *et al*[10,11] had proposed that the quantum noise and the pump noise are correlated under certain conditions. Since then, many researchers have taken into consideration the cross-correlation between the quantum noise and the pump noise when studying the statistical fluctuation properties, and the results obtained are in better agreement with the experimental results.[10−22] Zhou *et al*[23] considered the correlation between the real part and imaginary part of quantum noises, which attracted attention of many researchers. Later works further revealed that the locked phase of a laser is just caused by the cross-correlation between the real and imaginary parts of quantum noises.[24,25] In all the aforementioned papers, the pump noise and the quantum noise are both approximated as Gaussian white noise, and the cross-correlation between the two kinds of noises is also of Gaussian white noise. However, the power spectral density of white noises is constant, the truly white noise and white correlation do not exist, so it is necessary to consider the case driven by coloured noises with a coloured cross-correlation. Yet the negative diffusion problem caused by coloured noises has not been solved completely, and a laser system driven by coloured noises has been studied less. In 1994, using the unified coloured noise approximation (UCNA), Cao *et al*[16−19] studied the colour effect in a laser system driven by coloured noises, and the effect of quantum noises on the first-order-like phase transition. But, the UCNA is accurate only when the correlation time $\tau \to 0$ or $\tau \to \infty$. By far, there are few

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papers about the laser system with a coloured pump noise and a quantum noise with a cross-correlation.[26]

In this paper, we study the system of a single-mode laser driven by additive and multiplicative coloured noises with a coloured cross-correlation. The analytic expression of the stationary intensity distribution (SID) for the laser is derived for different correlation times. The influences of each stochastic parameter on the SID are studied, the effects of noise "colour" and the effect of the cross-correlation on the mean intensity, the variance and the skewness of laser intensity are also discussed. We found that there exist colourful phase transitions for the SID above a threshold and re-entrant phase transitions, which are induced by the "colour" of the additive noises. The study of the curves $< I >$-$\tau_i$, $\lambda_2(0)$-$\tau_i$, and $\lambda_3(0)$-$\tau_i$ ($i$=1,2,3) shows that not only the output intensity of the laser system increases with the additive noise self-correlation time $\tau_2$ and the cross-correlation time $\tau_3$, but also the quality of output of laser beams is optimized.

The rest of this paper is organized as follows. In Section 2, using the approximate Fokker–Planck equation for the general system with the correlated coloured multiplicative noise and the coloured additive noise we derived earlier, we calculate specifically the stationary probability distribution of the field amplitude in a cubic model, where the self-correlation time and cross-correlation time are different. In Section 3, the analytical expression of SID for laser intensity is obtained by using the Jacobian relation between the intensity $I$ and the field amplitude $x$. In Section 4, the calculation formulae of $< I >$, $\lambda_2(0)$, and $\lambda_3(0)$ are presented. In Section 5, we discuss in detail the SID, the mean intensity $< I >$, the variance $\lambda_2(0)$, and the skewness $\lambda_3(0)$, and a lot of new phenomena are summarized.

## 2. The stationary probability distribution of field amplitude

The stochastic process of the laser field amplitude is given by the following Langevin equation:[5,6]

$$\dot{x} = a_0 x - Ax^3 + \frac{p}{x} + x\xi(t) + \eta(t), \qquad (1)$$

where $a_0$ is the net gain coefficient, $A$ denotes the self-saturation coefficient, $p$ represents the strength of quantum noise. $x = \sqrt{I}$, $I$ is the intensity of laser. $\xi(t)$ and $\eta(t)$ represent the multiplicative noise and additive noise, respectively, whose statistical properties are

$$< \xi(t) >=< \eta(t) >= 0,$$

$$\gamma_{11}(t,t') = < \xi(t)\xi(t') > = \frac{p'}{\tau_1}\exp\left(-\frac{|t-t'|}{\tau_1}\right),$$

$$\gamma_{22}(t,t') = < \eta(t)\eta(t') > = \frac{p}{\tau_2}\exp\left(-\frac{|t-t'|}{\tau_2}\right),$$

$$\gamma_{12}(t,t') = < \xi(t)\eta(t') > = \frac{\lambda\sqrt{pp'}}{\tau_3}\exp\left(-\frac{|t-t'|}{\tau_3}\right),$$

$$\gamma_{21}(t,t') = < \eta(t)\xi(t') > = \gamma_{12}(t,t').$$

Here the self-correlation time $\tau_1$, $\tau_2$, and cross-correlation time $\tau_3$ of $\xi(t)$ and $\eta(t)$ can be different values.

By using the approximate method developed by the present authors in Refs.[26, 27], the approximate Fokker–Planck equation corresponding to Eq.(1) can be written as

$$\frac{\partial}{\partial t}Q(x,t) = -\frac{\partial}{\partial x}V(x)Q(x,t) + \frac{\partial^2}{\partial x^2}G(x)Q(x,t), \qquad (2)$$

where

$$V(x) = a_0 x - Ax^3 + \frac{p}{x} + \frac{p'}{1+2a_0\tau_1}x + \frac{\lambda\sqrt{pp'}}{1+2a_0\tau_3},$$

$$G(x) = \frac{p'}{1+2a_0\tau_1}x^2 + \frac{2\lambda\sqrt{pp'}}{1+2a_0\tau_3}x + \frac{p}{1+2a_0\tau_2},$$

and $1+2a_0\tau_i > 0$ ($i$=1,2,3).

When $\dfrac{\lambda^2(1+2a_0\tau_1)(1+2a_0\tau_2)}{(1+2a_0\tau_3)^2} > 1$, the stationary probability distribution of the field amplitude is obtained from Eq.(2),[28]

$$Q_{\rm st}(x) = N[G(x)]^{a-1}x^{1+2a_0\tau_2} \times \exp\left[-\frac{\beta}{2}Ax^2 + \frac{2A\beta\lambda\sqrt{pp'}}{1+2a_0\tau_3}x + \left(\gamma - \frac{2\lambda\sqrt{pp'}}{1+2a_0\tau_3}a\right)\frac{1}{\chi_0} \times \ln\frac{2x/\beta + 2\lambda\sqrt{pp'}/(1+2a_0\tau_3) - \chi_0}{2x/\beta + 2\lambda\sqrt{pp'}/(1+2a_0\tau_3) + \chi_0}\right], \qquad (3)$$

where

$$\beta = \frac{1+2a_0\tau_1}{p'},$$

$$a = \frac{\beta}{2}\left[a_0 + \frac{pA(1+2a_0\tau_1)}{p'(1+2a_0\tau_2)} - \frac{2a_0\tau_2 p'}{1+2a_0\tau_1} - 4A\lambda^2\frac{p}{p'}\left(\frac{1+2a_0\tau_1}{1+2a_0\tau_3}\right)^2\right],$$

$$\gamma = -\frac{\lambda\sqrt{pp'}(1+4a_0\tau_2)}{1+2a_0\tau_3} - \frac{2A\lambda p(1+2a_0\tau_2)^2\sqrt{pp'}}{(1+2a_0\tau_1)(1+2a_0\tau_3)},$$

$$\chi_0 = \sqrt{4pp'\left[\left(\frac{\lambda}{1+2a_0\tau_3}\right)^2 - \frac{1}{(1+2a_0\tau_1)(1+2a_0\tau_2)}\right]}.$$

When $\dfrac{\lambda^2(1+2a_0\tau_1)(1+2a_0\tau_2)}{(1+2a_0\tau_3)^2} < 1,$

$$Q_{\rm st}(x) = N[G(x)]^{a-1}x^{1+2a_0\tau_2} \times \exp\left[-\frac{\beta}{2}Ax^2 + \frac{2A\beta\lambda\sqrt{pp'}}{1+2a_0\tau_3}x + \left(\gamma - \frac{2\lambda\sqrt{pp'}}{1+2a_0\tau_3}a\right)\frac{2}{\chi_1} \times \arctan\frac{2x/\beta + 2\lambda\sqrt{pp'}/(1+2a_0\tau_3)}{\chi_1}\right], \quad (4)$$

where

$$\chi_1 = \sqrt{4pp'\left[\frac{1}{(1+2a_0\tau_1)(1+2a_0\tau_2)} - \left(\frac{\lambda}{1+2a_0\tau_3}\right)^2\right]}.$$

## 3. The SID of a single-mode laser

Since $x = \sqrt{I}$, according to the Jacobian relation between $Q(x)$ and the SID $P_{\rm st}(I)$, we have

$$P_{\rm st}(I) = Q_{\rm st}(x)\frac{{\rm d}x}{{\rm d}I}. \quad (5)$$

Thus, the SID can be derived from the stationary probability distribution of the field amplitude. Two cases of the SID are given as follows:

When $\dfrac{\lambda^2(1+2a_0\tau_1)(1+2a_0\tau_2)}{(1+2a_0\tau_3)^2} > 1,$

$$P_{\rm st}(I) = N[G(\sqrt{I})]^{a-1}I^{a_0\tau_2} \times \exp\left[-\frac{\beta}{2}AI + \frac{2A\beta\lambda\sqrt{pp'I}}{1+2a_0\tau_3} + \left(\gamma - \frac{2\lambda\sqrt{pp'}}{1+2a_0\tau_3}a\right)\frac{1}{\chi_0} \times \ln\frac{2\sqrt{I}/\beta + 2\lambda\sqrt{pp'}/(1+2a_0\tau_3) - \chi_0}{2\sqrt{I}/\beta + 2\lambda\sqrt{pp'}/(1+2a_0\tau_3) + \chi_0}\right]. \quad (6)$$

When $\dfrac{\lambda^2(1+2a_0\tau_1)(1+2a_0\tau_2)}{(1+2a_0\tau_3)^2} < 1,$

$$P_{\rm st}(I) = N[G(\sqrt{I})]^{a-1}I^{a_0\tau_2} \times \exp\left[-\frac{\beta}{2}AI + \frac{2A\beta\lambda\sqrt{pp'I}}{1+2a_0\tau_3} + \left(\gamma - \frac{2\lambda\sqrt{pp'}}{1+2a_0\tau_3}a\right)\frac{2}{\chi_1} \times \arctan\frac{2\sqrt{I}/\beta + 2\lambda\sqrt{pp'}/(1+2a_0\tau_3)}{\chi_1}\right]. \quad (7)$$

## 4. The mean, the variance, and the skewness of the laser intensity

At the $n$th moment, the laser intensity $I$ is given by the following integral:

$$< I^n > = \int_0^\infty I^n P_{\rm st}(I){\rm d}I. \quad (8)$$

The mean, the variance, and the skewness of the laser intensity can be numerically integrated from Eq.(8). The mean of the laser intensity reads

$$< I > = \int_0^\infty I P_{\rm st}(I){\rm d}I. \quad (9)$$

The variance, and the skewness of the laser intensity are defined as

$$\lambda_2(0) = \frac{< I^2 >}{< I >^2} - 1,$$
$$\lambda_3(0) = \frac{< I^3 >}{< I >^3} - 3\lambda_2(0) - 1. \quad (10)$$

In the next section, the effect of noise "colour" and the effect of the cross-correlation on the SID and the first three moments of the laser above the threshold are analysed thoroughly. Many novel phenomena are found.

# 5. Conclusion and discussion

### 5.1. $P_{\rm st}(I)$-$I$ curves with $\lambda$=0

According to Eqs.(6) and (7), the curves of the SID $P_{\rm st}(I)$ versus laser intensity $I$ are plotted in Figs.1 and 2. The two figures show that, if a cross-correlation does not exist ($\lambda$=0), no matter whatever the other parameters are adjusted to, the SID will still keep a single extreme, and phase transition will not appear. For example, when the self-correlation time $\tau_1$ of the multiplicative noise increases, the peak value of the SID increases, and the position of the peak shifts to the right-hand side (see Fig.1(a)). When the strength $p'$ of the multiplicative noise increases, the peak value of the SID also increases, but the position of the peak shifts to the left-hand side (see Fig.1(b)). Taking different values of the additive noise self-correlation time $\tau_2$, and the strength of the additive noise $P_{\rm st}$ does not fundamentally alter the curves (see Fig.2).

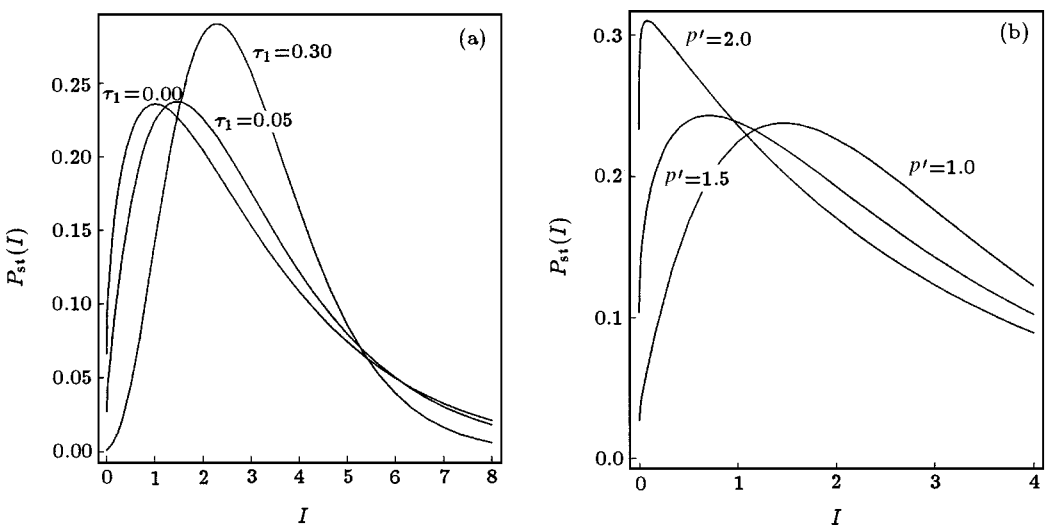



**Fig.1.** Steady-state laser intensity distribution functions with $a_0$=3, $p$=0.1, $\lambda$=0, $\tau_2$=0.03. (a) $p'$=1; (b) $\tau_1$=0.05.

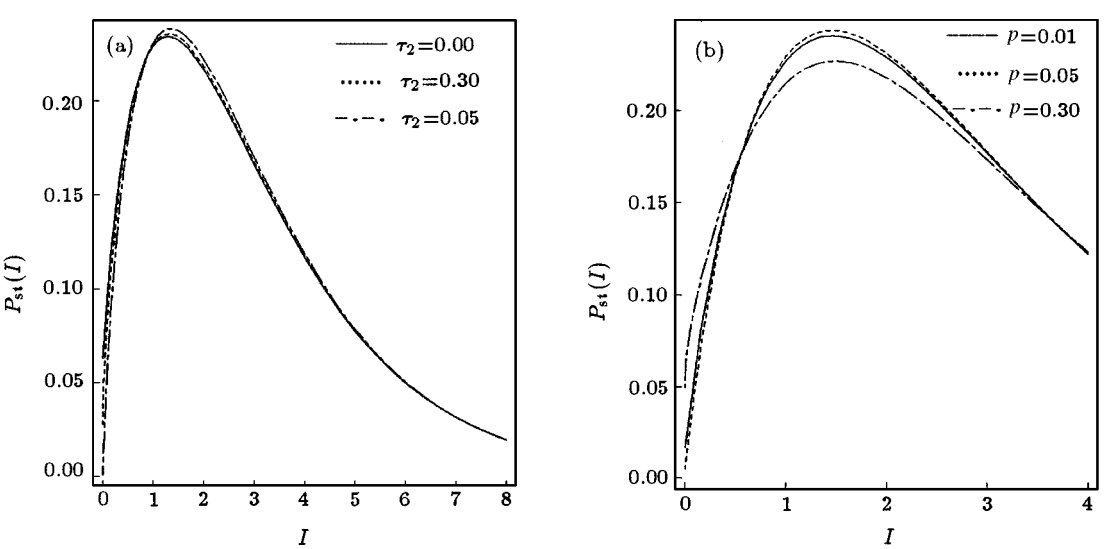



**Fig.2.** Steady-state laser intensity distribution functions with $a_0$=3, $p'$=1, $\lambda$=0, $\tau_1$=0.03. (a) $p$=0.1; (b) $\tau_2$=0.03.

### 5.2. $P_{\rm st}(I)$-$I$ curves for different values of $\lambda$

When there is a cross-correlation ($\lambda \neq 0$), we plot the curves of $P_{\rm st}$-$I$ for different values of $\lambda$ in Fig.3. The curves show that if any two correlation times among the three correlation times are not zero, namely $\tau_2 \neq 0$, $\tau_3 \neq 0$ or $\tau_1 \neq 0$, $\tau_2 \neq 0$ or $\tau_1 \neq 0$, $\tau_3 \neq 0$, then increasing $\lambda$, the curves will experience a transition from a single extremum → two extrema → monotonic state.

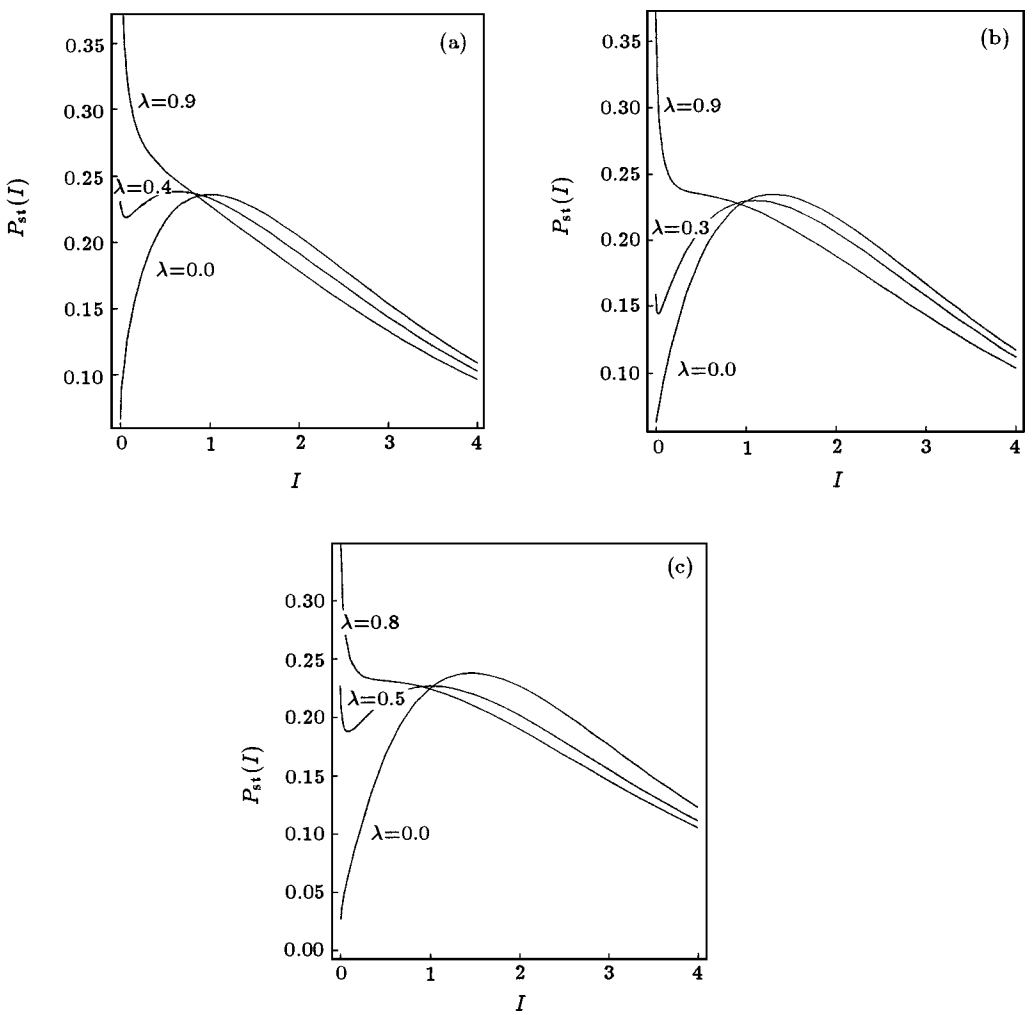



**Fig.3.** Steady-state laser intensity distribution functions with $a_0$=3, $p$=0.1, $p'$=1. (a) $\tau_1$=0, $\tau_2$=0.03, $\tau_3$=0.05; (b) $\tau_1$=0.03, $\tau_2$=0, $\tau_3$=0.05; (c) $\tau_1$=0.05, $\tau_2$=0.05, $\tau$=0.

### 5.3. The effect of noise correlation time $\tau_i$ on $P_{\rm st}(I)$

When there is a cross-correlation ($\lambda \neq 0$), we discuss the effect of correlation time $\tau_i$ ($i$=1,2,3) on the SID. In Fig.4(a), we assume $\tau_2$=0.03, $\tau_3$=0.05, and decrease $\tau_1$, while in Fig.4(b), we assume $\tau_1$=0.03, $\tau_2$=0.03, and decrease $\tau_3$, in this two cases, the $P_{\rm st}$-$I$ curves all exhibit the same phase transition (monotonic → two extrema → single extremum state), which is just contrary to the transition induced by increasing $\lambda$ (see Fig.3). In Fig.5, we let $\tau_1$=0.03, $\tau_3$=0.05, and increase $\tau_2$, it is seen that the SID exhibits four phases: two extrema → three extrema → single extremum → monotonic state. In Fig.6, we let $\tau_1$=0, $\tau_3$=0, and increase $\tau_2$, thus we have a re-entrant phase transition which is shown as monotonic → single extremum → monotonic state.

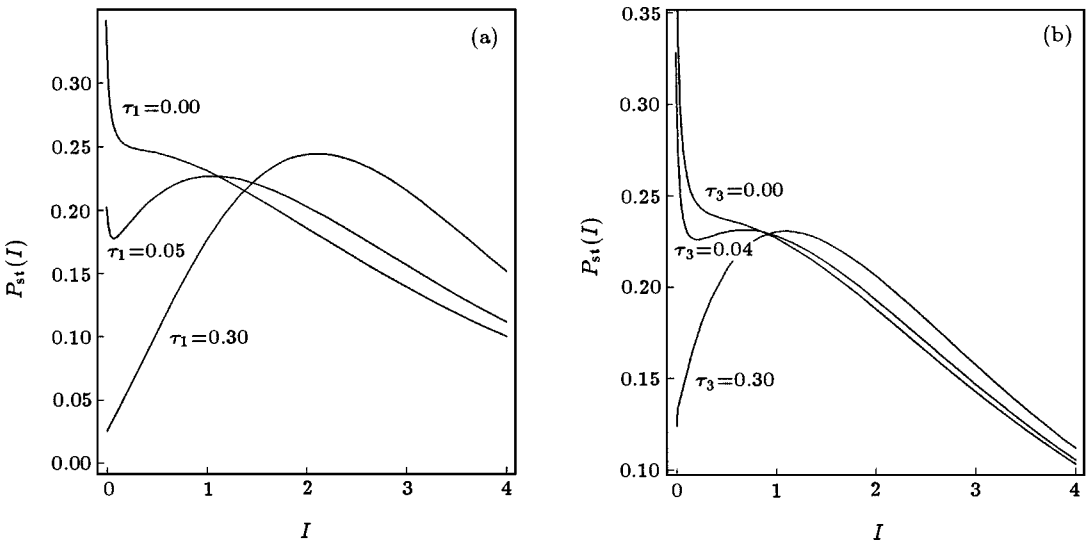



**Fig.4.** Steady-state laser intensity distribution functions with $a_0$=3, $p$=0.1, $p'$=1, $\lambda$=0.6, $\tau_2$=0.03. (a) $\tau_3$=0.05; (b) $\tau_1$=0.03.

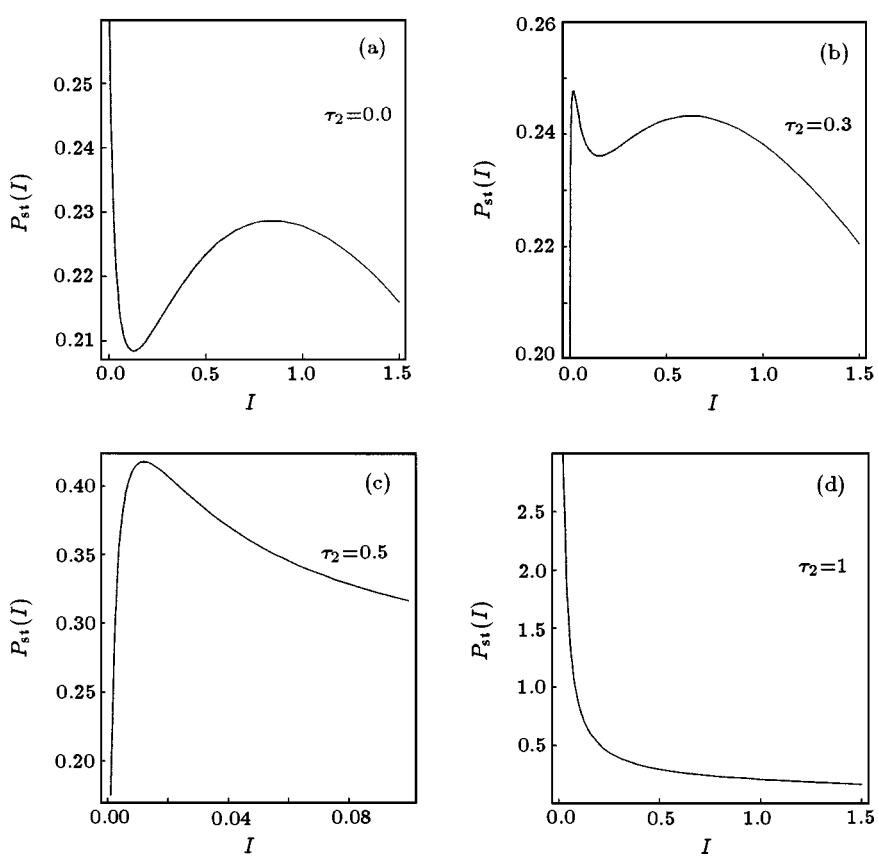



**Fig.5.** Steady-state laser intensity distribution functions with $a_0$=3, $p$=0.1, $p'$=1, $\lambda$=0.6, $\tau_1$=0.03, $\tau_3$=0.05.

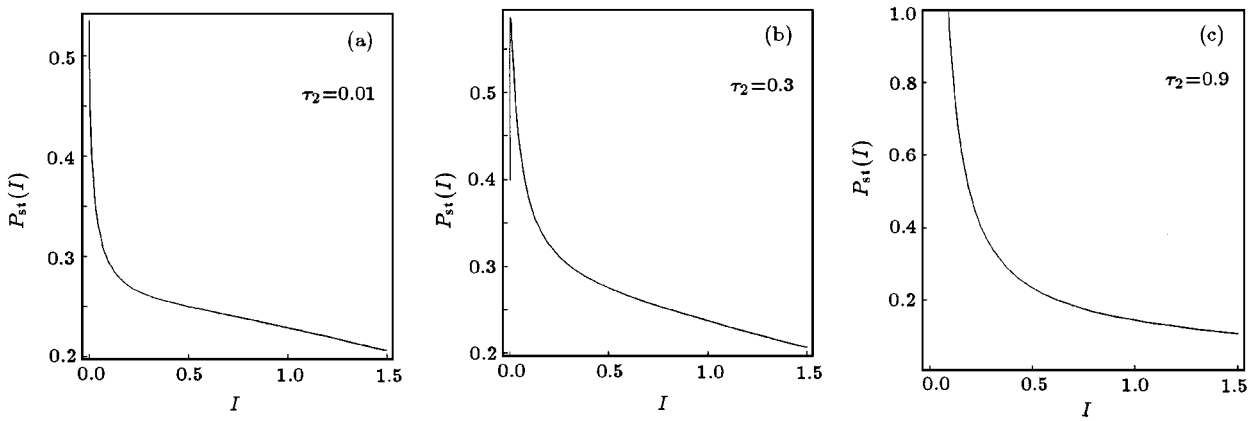



**Fig.6.** Steady-state laser intensity distribution functions with $a_0$=3, $p$=0.1, $p'$=1, $\lambda$=0.6, $\tau_1 = \tau_3$=0.

### 5.4. The effect of noise strength on $P_{st}(I)$

When correlation strength $\lambda$ is kept at a fixed value ($\lambda \neq 0$), phase transition also can be induced by increasing $p$ and $p'$, and phase transition is shown as: a single extremum $\rightarrow$ two extrema $\rightarrow$ monotonic state (see Fig.7). Phase transition induced by the strength $p'$ of multiplicative noise is just contrary to that induced by self-correlation time $\tau_1$ of multiplicative noise (compared Fig.7(a) with Fig.4(a)). However, phase transition induced by the strength $p$ of the additive noise is not contrary to that induced by the self-correlation time $\tau_2$ of the additive noise (compared Fig.7(b) with Fig.(5)).

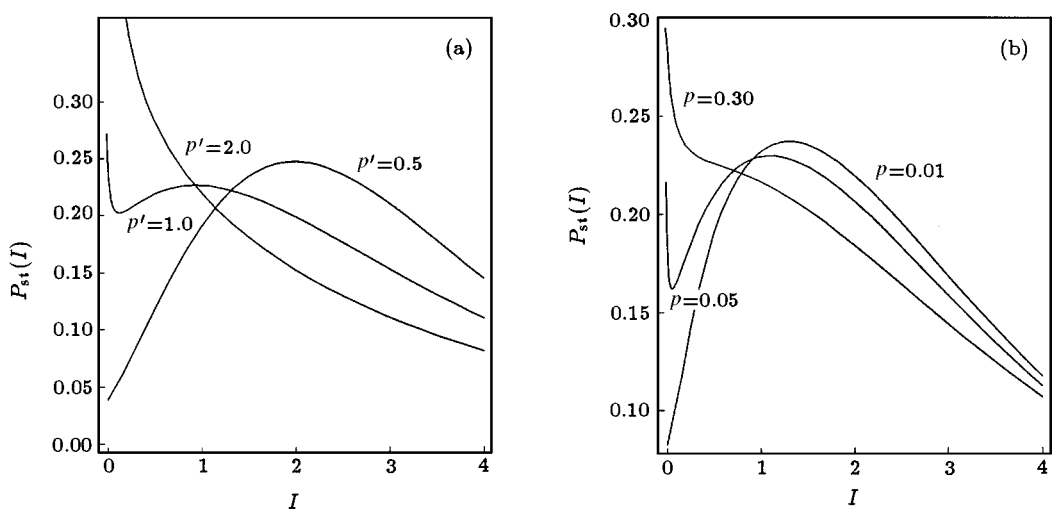



**Fig.7.** Steady-state laser intensity distribution functions with $a_0$=3, $\lambda$=0.6, $\tau_1$=0.05, $\tau_2$=0.03, $\tau_3$=0.01. (a) $p$=0.1, (b) $p'$=1.

From the above results, we can conclude that if there is a cross-correlation ($\lambda \neq 0$), phase transition can be induced. When $\tau_1 \neq 0$,$\tau_3 \neq 0$, the phase configuration (two extrema $\rightarrow$ three extrema $\rightarrow$ single extremum $\rightarrow$ monotonic) induced by $\tau_2$ is completely different from that induced by the other parameters. If there is not a cross-correlation ($\lambda$=0), and $\tau_1 = \tau_3$=0, as long as $\tau_2 \neq 0$, the "re-entrant phase transition" will exhibit by increasing $\tau_2$ (see Fig.6). So we can say that the "re-entrant phase transition" is a joint effect of the cross-correlation strength $\lambda$ and the "colour" of the additive noise. The additive noise performs a special function in the noise induced phase transition.

### 5.5. The effect of $\lambda$ and $\tau_i$ on $< I >$, $\lambda_2(0)$, and $\lambda_3(0)$

Here we discuss the effects of noise correlation strength $\lambda$ and the correlation time $\tau_i$ ($i$=1,2,3) on the $< I >$, $\lambda_2(0)$ and $\lambda_3(0)$. In Fig.8, the curves of $< I >$-$\tau_1$, $\lambda_2(0)$-$\tau_1$ and $\lambda_3(0)$-$\tau_1$ are depicted, respectively. The figures show that when $\lambda$=0, whether $\tau_2$ is zero or not, the mean output intensity will decrease monotonically with $\tau_1$ (see Figs.8(a) and 8(c)), that is, increasing $\tau_1$ is unfavourable to the output intensity. When $\lambda$=0.6 and $\tau_2$=$\tau_3$=0, the $< I >$-$\tau_1$ curve has a minimum, i.e. the mean output of the laser is

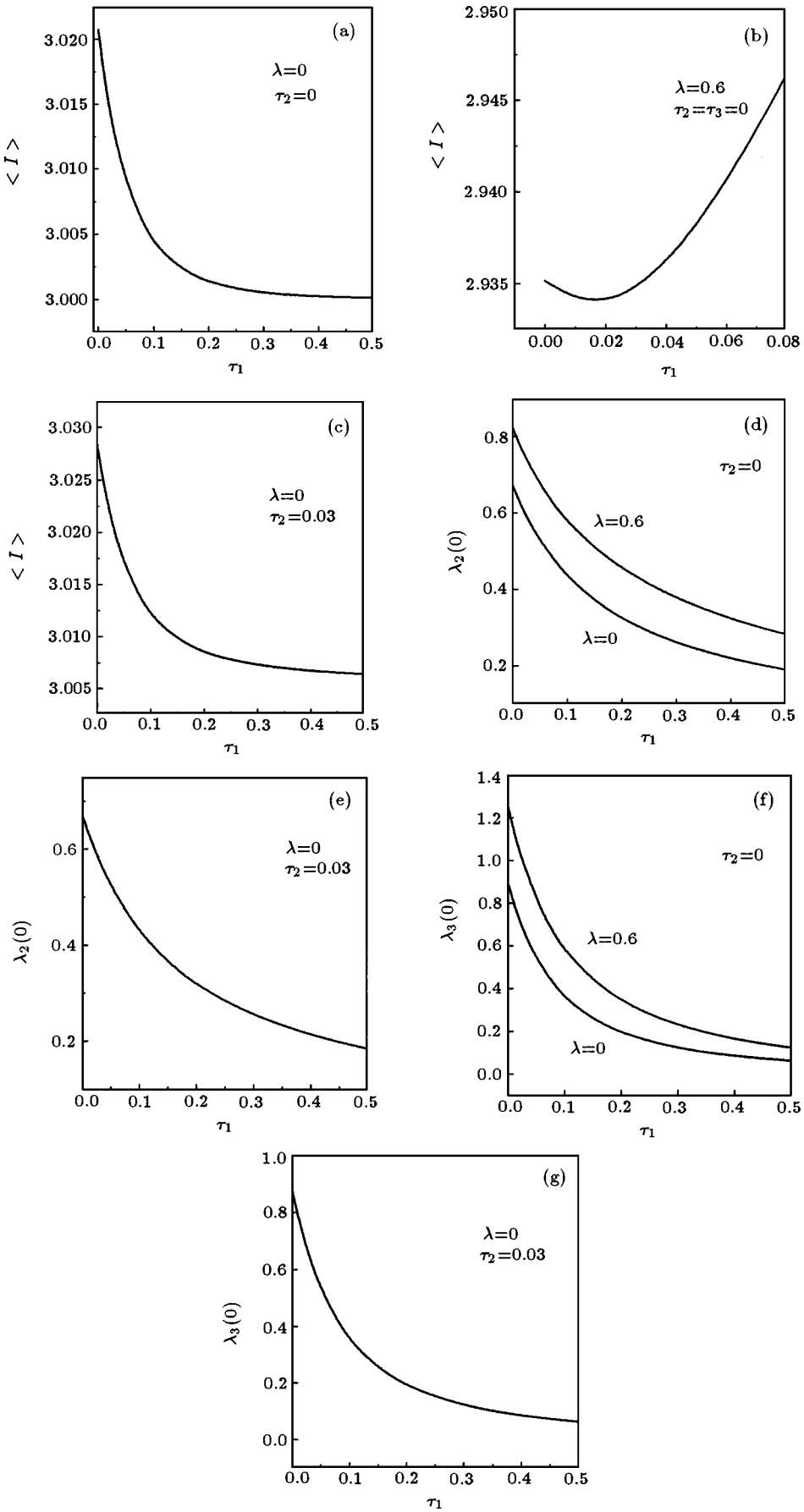



**Fig.8.** The mean normalized variance and skewness of the laser intensity as functions of the multiplicative noise self-correlation time $\tau_1$ with $a_0$=3, $p$=0.1, $p'$=1.

suppressed at the beginning, after passing the minimum ($\tau_1 > 0.018$), the mean $< I >$ increases with $\tau_1$ (see Fig.8(b)). $\lambda_2(0)$ and $\lambda_3(0)$ decrease with $\tau_1$ (see Figs.8(d)–(g)), which indicates the quality of the output of laser beams is optimized. In addition, Figs.8(d) and 8(f) also show that increasing $\lambda$, the entire curves of $\lambda_2(0)$-$\tau_1$ and $\lambda_3(0)$-$\tau_1$ go up. This means that the quality of output laser becomes poor. In Figs.9 and 10, the curves of $< I >$, $\lambda_2(0)$ and $\lambda_3(0)$ versus $\tau_2$ and $\tau_3$ are plotted, respectively. The figures show that the curves of $< I >$-$\tau_2$ and $< I >$-$\tau_3$ increase monotonically, and the curves of $\lambda_2(0)$-$\tau_2$, $\lambda_2(0)$-$\tau_3$, $\lambda_3(0)$-$\tau_2$ and $\lambda_3(0)$-$\tau_3$ decrease monotonically, which indicates that not only the output intensity increases with the additive noise self-correlation time $\tau_2$ and the cross-correlation time $\tau_3$, but also the quality of the output of laser beams is optimized.

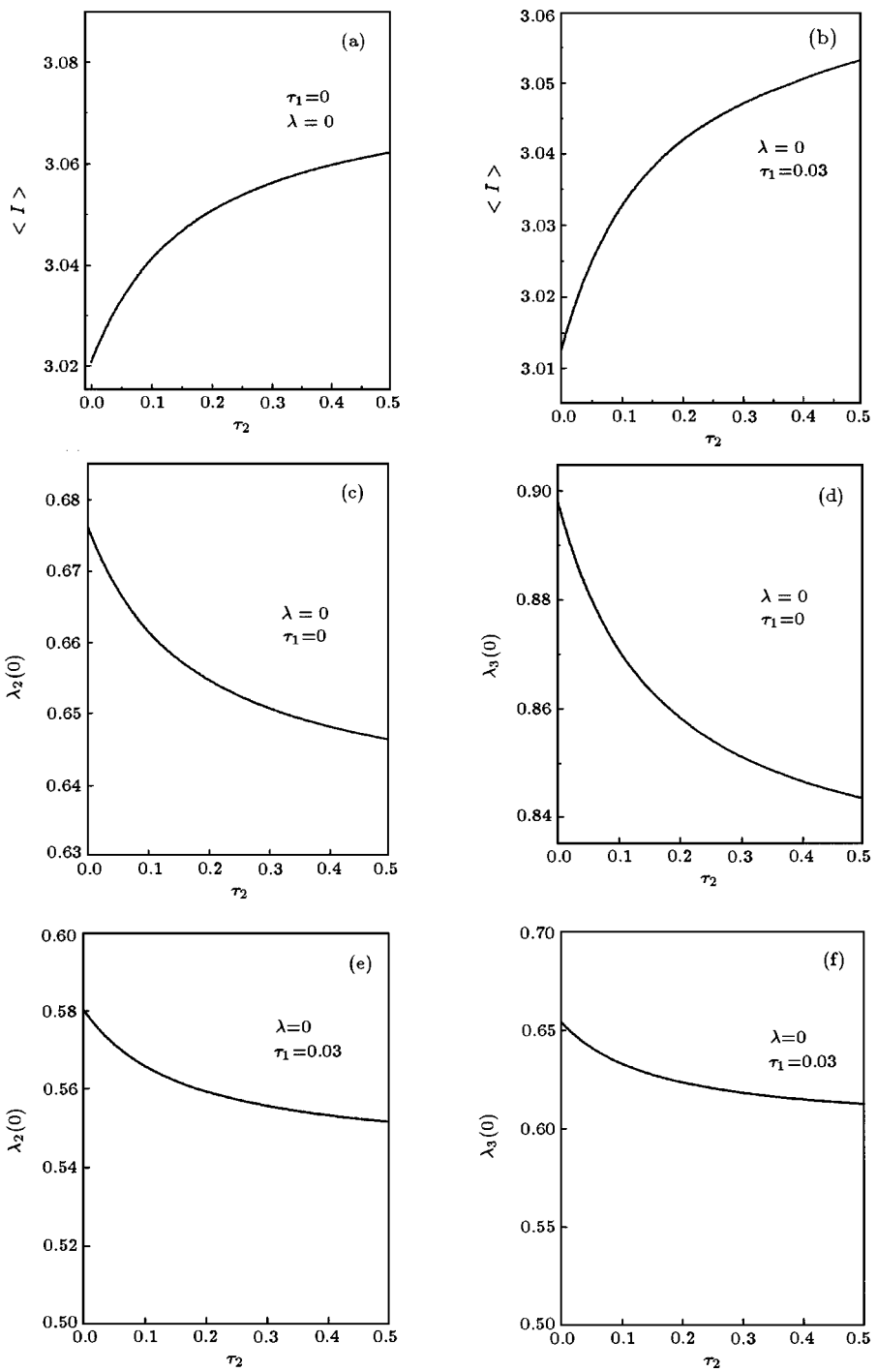



**Fig.9.** The mean normalized variance and skewness of the laser intensity as functions of the additive noise self-correlation time $\tau_2$ with $a_0$=3, $p$=0.1, $p'$=1.

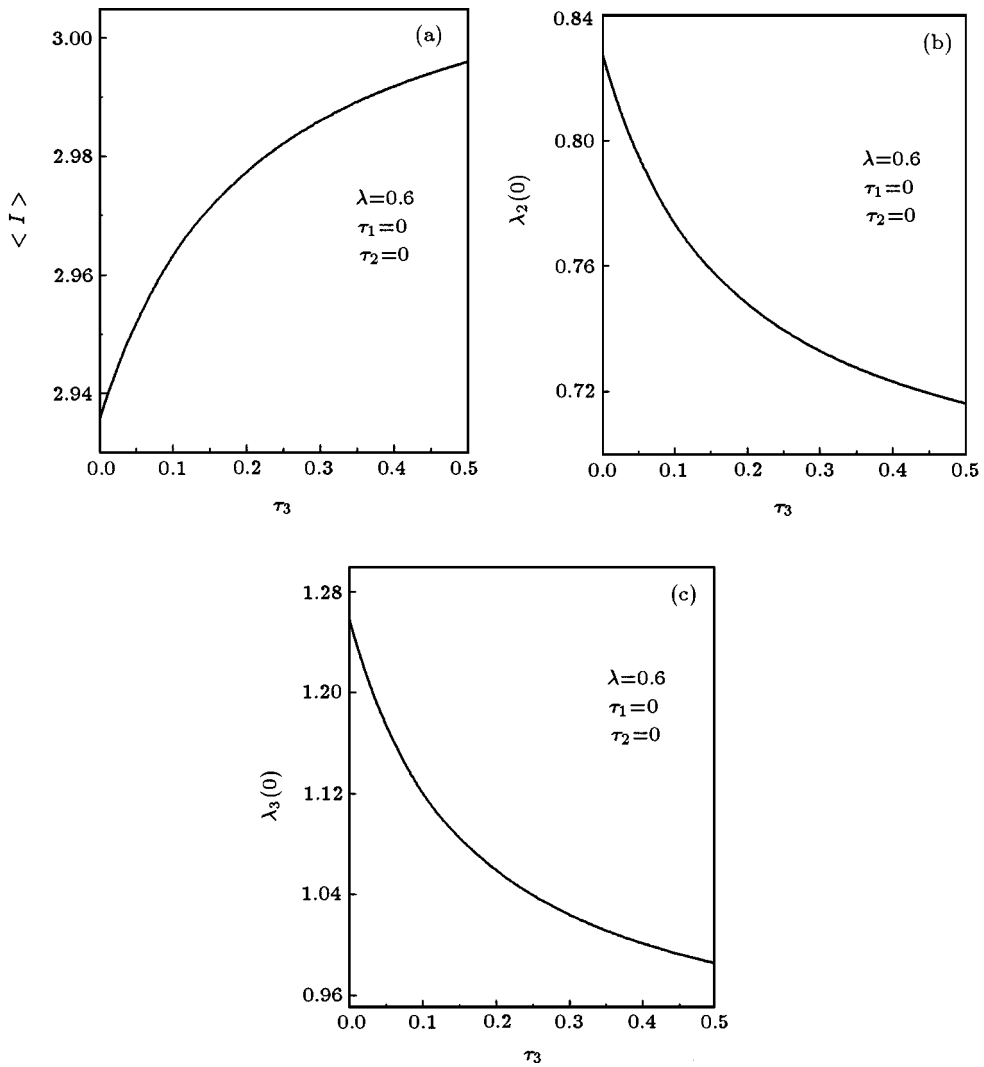



**Fig.10.** The mean normalized variance and skewness of the laser intensity as functions of the noise cross-correlation time $\tau_3$ with $a_0$=3, $p$=0.1, $p'$=1.

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