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Minimum signals in classical physics*

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The bandwidth theorem for Fourier analysis on any time-dependent classical signal is shown using the operator approach to quantum mechanics. Following discussions about squeezed states in quantum optics, the problem of minimum signals presented by a single quantity and its squeezing is proposed. It is generally proved that all such minimum signals, squeezed or not, must be real Gaussian functions of time.

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1. Introduction

All the studies on minimum uncertainty states and squeezed states of quantum optics and atomic-molecular physics[1−4] originate from the Heisenberg's uncertainty principle,[5,6] which states that for any given quantum state, the product of uncertainty amounts of two observables with noncommuting operators must be not smaller than a constant. For any minimum uncertainty state, the product reaches its lower limitation. The squeezed states belong to a special set of minimum states, in which the amount of one observable's uncertainty decreases but that of another observable will increase.[7−9]

No quantum fluctuation occurs in classical physics, but there exist some inequalities similar to the Heisenberg's uncertainty relations, such as the bandwidth theorem of Fourier analysis. This theorem is a quantitative expression of such an obvious fact that the duration of a signal is inverse proportional to its bandwidth of frequency. The time scaling transformation $t \mapsto t/T$ of a signal $f(t)$ must lead to a frequency scaling transformation $\omega \mapsto \omega/\Omega$ of its Fourier spectrum $F(\omega)$, and $\Omega = 1/T$.

In the early time of founding quantum mechanics, Schrödinger had studied the "most classical" coherent states of simple harmonic oscillators.[10] In those coherent states, the product of the uncertainties of position and momentum of a particle equals to $\hbar/2$, i.e. the lower limitation of the Heisenberg's uncertainty relation. If we introduce a "frequency operator" similar to the momentum operator in mathematics, the so-called Fourier bandwidth theorem can be easily proved following the way of quantum mechanics. This is the main topic of Section 2. The details of the proof provide us a mathematical foundation to study the problem of minimum signals and their squeezing. In Section 3, we present a general expression for minimum signals as a real function of time, and investigate some important properties of bandwidth. Finally, Section 4 is a short summary.

2. Fourier bandwidth theorem

Consider a signal presented by a single quantity of classical physics, such as the electric field of a linear polarization light, $y = f(t)$, and the Fourier spectrum of this time-dependent signal is defined as

$$F(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t. \tag{1}$$

The Fourier inverse transformation given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega \tag{2}$$

can be obtained by using the well known equality

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}(\omega' - \omega)t} \mathrm{d}t = \delta(\omega' - \omega). \tag{3}$$

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It means that the time-dependent signal corresponds, one to one, to its Fourier spectrum. In other words, $F(\omega)$ can express equivalently the signal $f(t)$. In many practical situations, the Fourier spectra are important characteristic functions of signals. Particularly, if the Fourier spectrum concentrates into a certain special region of frequency, the bandwidth will be an important parameter for describing the signal. The bandwidth $\Delta\omega$ is closely related to the signal's duration Δt. If a signal is shorter, the bandwidth of its spectrum is larger. In fact, we can prove rigorously the following bandwidth theorem.

Suppose that the signal is square-integrable, i.e.

$$\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t = \int_{-\infty}^{\infty} |F(\omega)|^2 \,\mathrm{d}\omega = C^2, \tag{4}$$

where C denotes a positive real constant, and the Parseval theorem of power spectrum has been applied to Eq.(4), then we have

$$\Delta\omega \cdot \Delta t \geq \frac{1}{2}, \tag{5}$$

where the bandwidth $\Delta\omega$ and duration Δt are defined as, respectively

$$(\Delta\omega)^2 \equiv \int_{-\infty}^{\infty} (\omega - \omega_0)^2 \left|\tilde{F}(\omega)\right|^2 \mathrm{d}\omega, \tag{6}$$

$$(\Delta t)^2 \equiv \int_{-\omega}^{\omega} (t - t_0)^2 \left|\tilde{f}(t)\right|^2 \mathrm{d}t. \tag{7}$$

Here the central frequency ω_0 and mean-time t_0 are defined in the traditional way

$$\omega_0 \equiv \int_{-\infty}^{\infty} \omega \left|\tilde{F}(\omega)\right|^2 \mathrm{d}\omega,$$

$$t_0 \equiv \int_{-\infty}^{\infty} t \left|\tilde{f}(t)\right|^2 \mathrm{d}t. \tag{8}$$

New functions $\tilde{f}$ and $\tilde{F}$ have been normalized, i.e. $\tilde{F}(\omega) \equiv F(\omega)/C$, $\tilde{f}(t) \equiv f(t)/C$. The time-dependent signal of a single quantity must be a real function of variable t, which implies that

$$f^*(t) = f(t), \qquad F^*(-\omega) = F(\omega). \tag{9}$$

Therefore, the central frequencies of signals must be zero, $\omega_0 = 0$, see Eq.(8); and the bandwidth defined in Eq.(6) is indeed the average value of square frequency, i.e.

$$(\Delta\omega)^2 = 2\int_0^{+\infty} \omega^2 \left|\tilde{F}(\omega)\right|^2 \mathrm{d}\omega. \tag{10}$$

It reflects the deviation of Fourier components from zero frequency.

To prove Eq.(5), we introduce a couple of Hermitian operators $\hat{\xi}$ and $\hat{\zeta}$, which satisfy the simple commutation relation

$$[\hat{\xi},\ \hat{\zeta}] = \mathrm{i}. \tag{11}$$

In the t-representation, they can be written as

$$\hat{\xi} = \mathrm{i}\frac{\partial}{\partial t}, \qquad \hat{\zeta} = t. \tag{12}$$

According to Eq.(1), all the average quantities defined by Eqs.(6)–(8) can be expressed as expectation values of corresponding operators about the "normalized wavefunction" $\tilde{f}(t)$ as follows:

$$\omega_0 = \int_{-\infty}^{\infty} \tilde{f}^*(t)\hat{\xi}\tilde{f}(t)\mathrm{d}t,$$

$$t_0 = \int_{-\infty}^{\infty} \tilde{f}^*(t)\hat{\zeta}\tilde{f}(t)\mathrm{d}t, \tag{13}$$

$$(\Delta\omega)^2 = \int_{-\infty}^{\infty} \tilde{f}^*(\Delta\hat{\xi})^2\tilde{f}\mathrm{d}t,$$

$$(\Delta t)^2 = \int_{-\infty}^{\infty} \tilde{f}^*(\Delta\hat{\zeta})^2\tilde{f}\mathrm{d}t, \tag{14}$$

where we introduce two new Hermitian operators $\Delta\hat{\xi} \equiv \hat{\xi} - \omega_0$ and $\Delta\hat{\zeta} \equiv \hat{\zeta} - t_0$. For any complex number λ, the operator $\Delta\hat{\xi} + \mathrm{i}\lambda\Delta\hat{\zeta}$ has a conjugate operator $\Delta\hat{\xi} - \mathrm{i}\lambda^*\Delta\hat{\zeta}$. So we can write down the following inequality:

$$\int_{-\infty}^{\infty} \tilde{f}^*(t)(\Delta\hat{\xi} - \mathrm{i}\lambda^*\Delta\hat{\zeta})(\Delta\hat{\xi} + \mathrm{i}\lambda\Delta\hat{\zeta})\tilde{f}(t)\mathrm{d}t \geq 0. \tag{15}$$

Simple calculation yields[11]

$$\lambda^*\lambda(\Delta t)^2 - \mathrm{Re}(\lambda) + (\Delta\omega)^2 - \mathrm{Im}(\lambda)\int_{-\infty}^{\infty} \tilde{f}^*(\Delta\hat{\xi}\Delta\hat{\zeta} + \Delta\hat{\zeta}\Delta\hat{\xi})\tilde{f}\mathrm{d}t \geq 0. \tag{16}$$

Here Re(λ)and Im(λ) denotes the real and imaginary parts of the complex number λ. Because the second-order polynomial of complex variable λ on the left-hand side of Eq.(16) must be non-negative, it is required that

$$(\Delta\omega)^2\cdot(\Delta t)^2\geq\frac{1}{4}+\left[\frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^{\infty}\tilde{f}^*\Delta\hat{\xi}\,\Delta\zeta\,\tilde{f}\mathrm{d}t\right)\right]^2. \tag{17}$$

If and only if

$$\left(\Delta\hat{\xi}+\mathrm{i}\lambda\Delta\hat{\zeta}\right)f(t)=0, \tag{18}$$

the equality in Eq.(17) holds. Noticing that the square term on the right-hand side of Eq.(17) is positive, then the Fourier bandwidth theorem (5) is proved.

3. Squeezing of signals

The time-dependent signals satisfying the equality of Eq.(5) have reached the minimum limitation of product $\Delta\omega\Delta t$. As far as we know, there is no record about such a problem in the literature. For convenience, we refer to such signals as minimum signals, following the nomination of minimum uncertainty states in quantum mechanics.[5−7] Eq. (18) is only a necessary condition for minimum signals. In addition, a minimum signal must make the square term on the right-hand side of Eq.(17) be zero, i.e.

$$\int_{-\infty}^{\infty}f^*(t)\left(\Delta\hat{\xi}\Delta\hat{\zeta}+\Delta\hat{\zeta}\Delta\hat{\xi}\right)f(t)\mathrm{d}t=0. \tag{19}$$

Combining Eqs.(18) and (19), it can be shown that the time-dependent functions of minimum signals must be an eigenfunction of the operator

$$\hat{G}_\lambda\equiv\hat{\xi}+\mathrm{i}\lambda\hat{\zeta}, \tag{20}$$

and here $\lambda^*=\lambda$ is a non-zero real number. If $\lambda=0$, the eigenfunctions $f(t)=\exp(\mathrm{i}\omega t)$ of operator $\hat{G}_0=\hat{\xi}$ describes some fictitious signals of infinite-long time duration, and cannot be square-integrable. Later in this paper, we will point out that the real number λ must be non-negative. In other words, the sufficient and necessary condition for minimum signals is that the time-dependent functions of them are eigenfunction of the operator $\hat{G}_\lambda\equiv\hat{\xi}+\mathrm{i}\lambda\hat{\zeta}$ with positive real parameter λ.

The operator technique of quantum mechanics is powerful for determining the minimum signals. As $\lambda>0$, we can define the couple of destruction and creation operators

$$\hat{a}\equiv\frac{1}{\sqrt{2\lambda}}\hat{\xi}+\mathrm{i}\sqrt{\frac{\lambda}{2}}\hat{\zeta},$$

$$\hat{a}^\dagger\equiv\frac{1}{\sqrt{2\lambda}}\hat{\xi}-\mathrm{i}\sqrt{\frac{\lambda}{2}}\hat{\zeta}. \tag{21}$$

They satisfy the simple Bosonic commutation relation $[\hat{a}\ ,\ \hat{a}^\dagger]=1$. Comparing Eq.(20) with Eq.(21), it is obvious that the minimum signals must be an eigenfunction of the destruction operator $\hat{a}$, i.e. the Glauber's squeezing coherent states[5,6]

$$|\alpha,\lambda\rangle=\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{+\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle_\lambda. \tag{22}$$

Here α can be arbitrary complex number, the ket vector $|n\rangle$ denotes the eigenstate of "number of particle" operator $\hat{n}\equiv\hat{a}^\dagger\hat{a}$, and $\hat{a}\,|n\rangle=\sqrt{n}\,|n-1\rangle$, $\hat{a}^\dagger\,|n\rangle=\sqrt{n+1}\,|n+1\rangle$, $\hat{a}\,|\alpha,\lambda\rangle=\alpha\,|\alpha,\lambda\rangle$. With the help of Eq.(21), the "number of particle" operator can be written as

$$\hat{n}=-\frac{1}{2\lambda}\frac{\partial^2}{\partial t^2}+\frac{1}{2}\lambda t^2-\frac{1}{2}, \tag{23}$$

which is similar to the Hamiltonian of one-dimensional simple harmonic oscillators with mass $m=\lambda$ and spring-constant $k=\lambda$. Therefore, the "wavefunction" in t-representation can be directly written as[6]

$$\langle t|n\rangle_\lambda=N_n\lambda^{1/2}\exp\left(-\frac{\lambda t^2}{2}\right)\mathrm{H}_n(\lambda^{1/2}t), \tag{24}$$

where $\mathrm{H}_n(t)$ denotes the nth Hermitian polynomial, which satisfy the iterative relation $\mathrm{H}_{n+1}(t)=2t\mathrm{H}_n(t)-2n\mathrm{H}_{n-1}(t)$, and $\mathrm{H}_0(t)=1$, $\mathrm{H}_1(t)=2t$, and the normalization constant $N_n=2^{-n/2}(n!)^{-1/2}\pi^{-1/4}$. Therefore, the general expression of the time-dependent function of minimum signals $\tilde{f}_{\alpha\lambda}(t)=\langle t|\alpha,\lambda\rangle$ can be obtained by inserting Eq.(24) into Eq.(22), i.e.

$$\tilde{f}_{\alpha\lambda}(t)=\pi^{-1/4}\lambda^{1/2}\exp\left(-\frac{\lambda t^2+|\alpha|^2}{2}\right)\times\sum_{n=0}^{+\infty}(n!)^{-1}2^{-n/2}\alpha^n\mathrm{H}_n(\lambda^{1/2}t). \tag{25}$$

To keep the functions of time-varying signals real, the parameter α should be real. Then the above equation can be simplified by utilizing the generating function of Hermitian polynomial[6,10]

$$\tilde{f}_{\alpha\lambda}(t)=\pi^{-1/4}\lambda^{1/4}\exp\left[-\frac{\lambda(t-t_0)^2}{2}\right], \tag{26}$$

where $t_0 = 2^{1/2}\lambda^{-1/2}\alpha$. It implies that the time-dependent functions of any minimum signals must be real Gaussian, which have Fourier spectra as below:

$$\tilde{F}_\lambda(\omega) = (\pi\lambda)^{-1/4}\exp\left[-\frac{\omega^2}{2\lambda} + \mathrm{i}\omega t_0\right] \qquad (27)$$

and

$$\Delta\omega = \sqrt{\frac{\lambda}{2}},$$
$$\Delta t = \frac{1}{\sqrt{2\lambda}}. \qquad (28)$$

The positive real number λ plays the role of squeezing parameter, and $\Delta\omega = \lambda\Delta t$. It is worthwhile to point out that the Gaussian functions defined in Eqs.(25) and (26) are familiar to physicists and mathematicians. Not only it has been used to describe the Schrödinger coherent states,[10] but also appears in recent study on multi-photon squeezed states of quantum optics.[3]

Supposing $\lambda < 0$, we can prove, by combining Eqs.(20) and (21), that the possible minimum signals should be eigenstates of creation operator $\hat{a}^\dagger$. However, it is well known that any eigenket vector of bosonic creation operator cannot be normalized,[12,13] so there should be no minimum signal for the negative parameter λ, as we mentioned in the paragraph next to Eq.(20).

4. Summary

We propose the concept of minimum signals and study the relevant problem of squeezing signals. It is shown rigorously that any signal presented by single classical quantity must be real Gaussian function of time. We expect that this work will inspire further research on the minimum signals of optics and acoustics. The Fourier spectrum of real Gaussian signal is still Gaussian with central frequency of zero. We guess it is a challenge to realize an optical (or electromagnetic) minimum signal with zero central frequency. The confinement of zero central frequency arises from the requirement that the signals are presented by single quantity. In a more general scheme of classical physics, the minimum signals are represented by two physical quantities, and can possess non-zero central frequency.

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