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Form invariance for systems of generalized classical mechanics*

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This paper presents a form invariance of canonical equations for systems of generalized classical mechanics. According to the invariance of the form of differential equations of motion under the infinitesimal transformations, this paper gives the definition and criterion of the form invariance for generalized classical mechanical systems, and establishes relations between form invariance, Noether symmetry and Lie symmetry. At the end of the paper, an example is given to illustrate the application of the results.

Keywords: system of generalized classical mechanics, canonical equation, form invariance, Lie symmetry, Noether symmetry

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1. Introduction

The method of symmetry or invariance is a modern method of research in the conserved quantities of mechanical system, and there are mainly the Noether symmetry[1−4] and the Lie symmetry.[3−8] Recently, Mei[9] presented an invariance for mechanical system which is called form invariance. The form invariance is a symmetry which makes the dynamical functions appearing in the differential equations of motion still satisfy the former equations after undergoing infinitesimal transformations. The form invariance is different from the Noether symmetry, and also different from the Lie symmetry. Some results[9−15] have been obtained in the study of form invariance. This paper further studies the form invariance for the system of generalized classical mechanics in high-dimensional extended phase space, discusses the relations between the form invariance and the Noether symmetry and Lie symmetry, and gives the condition under which a form invariance can lead to a conserved quantity and its form.

2. Form invariance of differential equation of motion of a system

Consider a system of generalized classical mechanics with n degrees of freedom. Suppose the configuration of the system is determined by n generalized coordinates $q^i (i = 1, \cdots, n)$, and the Lagrangian of the system is

$$L = L(t, q^i_{(0)}, q^i_{(1)}, \cdots, q^i_{(w)}) \quad (i = 1, \cdots, n), \tag{1}$$

where

$$q^i_{(j)} = \frac{\mathrm{d}^j}{\mathrm{d}t^j} q^i(t) \quad (i = 1, \cdots, n; j = 0, 1, \cdots, \omega). \tag{2}$$

The generalized Euler–Lagrange equations of the system are[2]

$$\sum_{j=0}^{\omega} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial q^i_{(j)}} = 0 \quad (i = 1, \cdots, n). \tag{3}$$

Introduce the generalized momenta and the generalized Hamiltonian function,

$$p^{(s)}_i = p_{i/(s+1)} = \sum_{j=0}^{\omega-s-1} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j} \frac{\partial L}{\partial q^i_{(j+s+1)}}$$

$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1), \tag{4}$$

$$H(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}) = p^{(s)}_i q^i_{(s+1)} - L$$

$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1), \tag{5}$$

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and assume the Lagrangian of the system is regular, then the equations of motion of the system, Eq.(3), can be expressed in the canonical form as follows:[2]

$$\dot{q}^i_{(s)} = \frac{\partial H}{\partial p^{(s)}_i},$$

$$\dot{p}^{(s)}_i = -\frac{\partial H}{\partial q^i_{(s)}}$$

$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1) \tag{6}$$

In high-dimensional extended phase space, we take the infinitesimal transformations

$$\begin{aligned} &t^* = t + \Delta t, \\ &q^{i^*}_{(s)}(t^*) = q^i_{(s)}(t) + \Delta q^i_{(s)}, \\ &p^{(s)^*}_i(t^*) = p^{(s)}_i(t) + \Delta p^{(s)}_i \end{aligned} \tag{7}$$

and their expanded formulae

$$\begin{aligned} &t^* = t + \varepsilon\tau(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}), \\ &q^{i^*}_{(s)} = q^i_{(s)} + \varepsilon\xi^i_{(s)}(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}), \\ &p^{(i)^*}_i = p^{(s)}_i + \varepsilon\eta^{(s)}_i(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}), \end{aligned} \tag{8}$$

where ε is an infinitesimal parameter $\tau, \xi^i_{(s)}$ and $\eta^{(s)}_i$ are generators of the infinitesimal transformations. Introduce the infinitesimal generator vector

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \xi^i_{(s)}\frac{\partial}{\partial q^i_{(s)}} + \eta^{(s)}_i\frac{\partial}{\partial p^{(s)}_i} \tag{9}$$

and its first extended vector

$$\begin{aligned} X^{(1)} =& X^{(0)} + (\dot{\xi}^i_{(s)} - \dot{q}^i_{(s)}\dot{\tau})\frac{\partial}{\partial \dot{q}^i_{(s)}} \\ &+ (\dot{\eta}^{(s)}_i - \dot{p}^{(s)}_i\dot{\tau})\frac{\partial}{\partial \dot{p}^{(s)}_i}. \end{aligned} \tag{10}$$

Suppose that under the infinitesimal transformations (8), the generalized Hamiltonian function $H(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)})$ becomes $H^* = H(t^*, \boldsymbol{q}^*_{(s)}, \boldsymbol{p}^{(s)^*})$.

Definition 1 Under the infinitesimal transformations (8), if the form of the equations of motion (6) preserves an invariance, i.e. the following equations hold:

$$\dot{q}^i_{(s)} = \frac{\partial H^*}{\partial p^{(s)}_i}, \quad \dot{p}^{(s)}_i = -\frac{\partial H^*}{\partial q^i_{(s)}}$$

$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1), \tag{11}$$

then the invariance is called a form invariance of the equations of motion for systems of the generalized classical mechanics.

Expanding H^*, we have

$$H^* = H(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}) + \varepsilon X^{(0)}(H) + o(\varepsilon^2), \tag{12}$$

therefore, we obtain the following theorem.

Theorem 1 For a system of generalized classical mechanics expressed by Eqs.(6), if the infinitesimal generators τ, $\xi^i_{(s)}$ and $\eta^{(s)}_i$ satisfy the following relations:

$$\frac{\partial X^{(0)}(H)}{\partial p^{(s)}_i} = 0, \quad \frac{\partial X^{(0)}(H)}{\partial q^i_{(s)}} = 0$$

$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1), \tag{13}$$

then it is form invariant under the transformation (8).

Proof Substituting formula (12) into Eqs.(11), using Eqs.(6), and ignoring ε^2 and the higher infinitesimal terms, we will obtain Eqs.(13).

3. Form invariance and Noether symmetry

According to the Noether theory, if there exists a gauge function $G = G(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)})$, such that the infinitesimal generators τ and $\xi^i_{(s)}$ satisfy the generalized Noether identity,

$$p^{(s)}_i\dot{\xi}^i_{(s)} - \frac{\partial H}{\partial t}\tau - \frac{\partial H}{\partial q^i_{(s)}}\xi^i_{(s)} - H\dot{\tau} + \dot{G} = 0, \tag{14}$$

then the corresponding symmetry is a Noether symmetry of the generalized classical mechanical system (6). Therefore, we have

Theorem 2 For the generalized classical mechanical system (6), if the infinitesimal transformations (8) are the transformations of form invariance, and there exists a gauge function $G = G(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)})$ such that the infinitesimal generators τ and $\xi^i_{(s)}$ satisfy the generalized Noether identity (14), then the form invariance of the system is also a Noether symmetry; otherwise it is not a Noether symmetry.

4. Form invariance and Lie symmetry

The basic idea of the Lie method is to keep the differential equation of motions (6) invariant under infinitesimal transformations (8). Eqs.(6) are rewritten as

$$\begin{aligned} &\dot{q}^i_{(s)} = g^i_{(s)}(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}), \\ &\dot{p}^{(s)}_i = h^{(s)}_i(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}) \end{aligned}$$

$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1). \tag{15}$$

Definition 2 If the differential equations (15) remain invariant under the infinitesimal transformation (8), i.e. the following equations hold:

$$\frac{\mathrm{d}q^{i^*}_{(s)}}{\mathrm{d}t^*} = g^i_{(s)}(t^*, \boldsymbol{q}^*_{(s)}, \boldsymbol{p}^{(s)^*}),$$

$$\frac{\mathrm{d}p_i^{(s)^*}}{\mathrm{d}t^*} = h_i^{(s)}(t^*, \boldsymbol{q}^*_{(s)}, \boldsymbol{p}^{(s)^*})$$
$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1), \tag{16}$$

then the invariance is called the Lie symmetry of the generalized classical mechanical system (6).

Since

$$\frac{\mathrm{d}q^{i^*}_{(s)}}{\mathrm{d}t^*} = \frac{\mathrm{d}q^i_{(s)} + \varepsilon \mathrm{d}\xi^i_{(s)}}{\mathrm{d}t + \varepsilon \mathrm{d}\tau} = \dot{q}^i_{(s)} + \varepsilon(\dot{\xi}^i_{(s)} - \dot{q}^i_{(s)}\dot{\tau}) + o(\varepsilon^2), \tag{17}$$

$$\frac{\mathrm{d}p_i^{(s)^*}}{\mathrm{d}t^*} = \frac{\mathrm{d}p_i^{(s)} + \varepsilon \mathrm{d}\eta_i^{(s)}}{\mathrm{d}t + \varepsilon \mathrm{d}\tau} = \dot{p}_i^{(s)} + \varepsilon(\dot{\eta}_i^{(s)} - \dot{p}_i^{(s)}\dot{\tau}) + o(\varepsilon^2), \tag{18}$$

$$g^i_{(s)}(t^*, \boldsymbol{q}^*_{(s)}, \boldsymbol{p}^{(s)^*}) = g^i_{(s)}(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}) + \varepsilon X^{(0)}(g^i_{(s)}) + o(\varepsilon^2), \tag{19}$$

$$h_i^{(s)}(t^*, \boldsymbol{q}^*_{(s)}, \boldsymbol{p}^{(s)^*}) = h_i^{(s)}(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)}) + \varepsilon X^{(0)}(h_i^{(s)}) + o(\varepsilon^2), \tag{20}$$

we have

Theorem 3 For the generalized classical mechanical system (6), if the infinitesimal generators τ, $\xi^i_{(s)}$ and $\eta_i^{(s)}$ satisfy the following relations:

$$\dot{\xi}^i_{(s)} - \frac{\partial H}{\partial p_i^{(s)}}\dot{\tau} = X^{(0)}\left(\frac{\partial H}{\partial p_i^{(s)}}\right),$$
$$\dot{\eta}_i^{(s)} + \frac{\partial H}{\partial \dot{q}^i_{(s)}}\dot{\tau} = -X^{(0)}\left(\frac{\partial H}{\partial q^i_{(s)}}\right)$$
$$(i = 1, \cdots, n; s = 0, 1, \cdots, \omega - 1), \tag{21}$$

then the system (6) is Lie symmetric.

From formula (9), it is easy to see that

$$\frac{\partial X^{(0)}(H)}{\partial p_i^{(s)}} = X^{(0)}\left(\frac{\partial H}{\partial p_i^{(s)}}\right) + \frac{\partial \tau}{\partial p_i^{(s)}}\frac{\partial H}{\partial t} + \frac{\partial \xi^j_{(k)}}{\partial p_i^{(s)}}\frac{\partial H}{\partial q^j_{(k)}} + \frac{\partial \eta_j^{(k)}}{\partial p_i^{(s)}}\frac{\partial H}{\partial p_j^{(k)}}, \tag{22}$$

$$\frac{\partial X^{(0)}(H)}{\partial q^i_{(s)}} = X^{(0)}\left(\frac{\partial H}{\partial q^i_{(s)}}\right) + \frac{\partial \tau}{\partial q^i_{(s)}}\frac{\partial H}{\partial t} + \frac{\partial \xi^j_{(k)}}{\partial q^i_{(s)}}\frac{\partial H}{\partial q^j_{(k)}} + \frac{\partial \eta_j^{(k)}}{\partial q^i_{(s)}}\frac{\partial H}{\partial p_j^{(k)}}. \tag{23}$$

Therefore, we have

Theorem 4 For the generalized classical mechanical system (6), if the infinitesimal transformations (8) are the transformations of form invariance, and the infinitesimal generators τ, $\xi^i_{(s)}$ and $\eta_i^{(s)}$ satisfy the conditions

$$\dot{\xi}^i_{(s)} - \frac{\partial H}{\partial p_i^{(s)}}\dot{\tau} + \frac{\partial \tau}{\partial p_i^{(s)}}\frac{\partial H}{\partial t} + \frac{\partial \xi^j_{(k)}}{\partial p_i^{(s)}}\frac{\partial H}{\partial q^j_{(k)}} + \frac{\partial \eta_j^{(k)}}{\partial p_i^{(s)}}\frac{\partial H}{\partial p_j^{(k)}} = 0, \tag{24}$$

$$\dot{\eta}_i^{(s)} + \frac{\partial H}{\partial q^i_{(s)}}\dot{\tau} - \frac{\partial \tau}{\partial q^i_{(s)}}\frac{\partial H}{\partial t} - \frac{\partial \xi^j_{(k)}}{\partial q^i_{(s)}}\frac{\partial H}{\partial q^j_{(k)}} - \frac{\partial \eta_j^{(k)}}{\partial q^i_{(s)}}\frac{\partial H}{\partial p_j^{(k)}} = 0, \tag{25}$$

then the infinitesimal transformations (8) are also the transformations of Lie symmetry of the system.

5. Form invariance and conserved quantity

In general, a form invariance does not always lead to a conserved quantity, and the following theorem gives the condition under which a form invariance can lead to a conserved quantity and its form.

Theorem 5 For the generalized classical mechanical system (6), if the infinitesimal transformations (8) are the transformations of form invariance of the system, and there exists a gauge function $G = G(t, \boldsymbol{q}_{(s)}, \boldsymbol{p}^{(s)})$ such that they satisfy the structure equation

$$-H\dot{\tau} + \frac{\partial H}{\partial p_i^{(s)}}\eta_i^{(s)} + p_i^{(s)}\dot{\xi}^i_{(s)} - X^{(0)}(H) + \dot{G} = 0, \tag{26}$$

then the system has a conserved quantity of form invariance as follows:

$$I = -H\tau + p_i^{(s)}\xi^i_{(s)} + G = \text{constant}. \tag{27}$$

6. Example

Suppose the Lagrangian of a generalized classical mechanical system is

$$L = \frac{1}{2}\alpha_1(q^1_{(1)})^2 + \frac{1}{2}\alpha_2(q^1_{(2)})^2 \quad (\alpha_1 > 0, \alpha_2 > 0). \tag{28}$$

We study its form invariance.

The generalized momenta and the generalized Hamiltonian function are

$$p_1^{(0)} = \alpha_1 q^1_{(1)} - \alpha_2 \dot{q}^1_{(2)}, \quad p_1^{(1)} = \alpha_2 q^1_{(2)}, \tag{29}$$

$$H = p_1^0 q_{(1)}^1 + \frac{1}{2\alpha_2}(p_1^{(1)})^2 - \frac{1}{2}\alpha_1 (q_{(1)}^1)^2. \tag{30}$$

The differential equations of motion of the system are

$$\begin{aligned}
\dot{q}_{(0)}^1 &= \frac{\partial H}{\partial p_1^{(0)}} = q_{(1)}^1, \\
\dot{q}_{(1)}^1 &= \frac{\partial H}{\partial p_1^{(1)}} = \frac{1}{\alpha_2} p_1^{(1)}, \\
\dot{p}_1^{(0)} &= -\frac{\partial H}{\partial q_{(0)}^1} = 0, \\
\dot{p}_1^{(1)} &= -\frac{\partial H}{\partial q_{(1)}^1} = -p_1^{(0)} + \alpha_1 q_{(1)}^1.
\end{aligned} \tag{31}$$

Equations (13) give

$$\begin{aligned}
&\xi_{(1)}^1 + \frac{\partial \xi_{(1)}^1}{\partial p_1^{(0)}}(p_1^{(0)} - \alpha_1 q_{(1)}^1) \\
&+ \frac{\partial \eta_1^{(0)}}{\partial p_1^{(0)}} q_{(1)}^1 + \frac{\partial \eta_1^{(1)}}{\partial p_1^{(0)}} \cdot \frac{1}{\alpha_2} p_1^{(1)} = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \xi_{(1)}^1}{\partial p_1^{(1)}}(p_1^{(0)} - \alpha_1 q_{(1)}^1) + \frac{\partial \eta_1^{(0)}}{\partial p_1^{(1)}} q_{(1)}^1 \\
&+ \frac{1}{\alpha_2}\eta_1^{(1)} + \frac{\partial \eta_1^{(1)}}{\partial p_1^{(1)}} \cdot \frac{1}{\alpha_2} p_1^{(1)} = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \xi_{(1)}^1}{\partial q_{(0)}^1}(p_1^{(0)} - \alpha_1 q_{(1)}^1) \\
&+ \frac{\partial \eta_1^{(0)}}{\partial q_{(0)}^1} q_{(1)}^1 + \frac{\partial \eta_1^{(1)}}{\partial q_{(0)}^1} \cdot \frac{1}{\alpha_2} p_1^{(1)} = 0,
\end{aligned}$$

$$\begin{aligned}
&- \alpha_1 \xi_{(1)}^1 + \frac{\partial \xi_{(1)}^1}{\partial q_{(1)}^1}(p_1^{(0)} - \alpha_1 q_{(1)}^1) \\
&+ \eta_1^{(0)} + \frac{\partial \eta_1^{(0)}}{\partial q_{(1)}^1} q_{(1)}^1 + \frac{\partial \eta_1^{(1)}}{\partial q_{(1)}^1} \cdot \frac{1}{\alpha_2} p_1^{(1)} = 0.
\end{aligned} \tag{32}$$

Equations (32) have a solution as follows:

$$\tau = -1, \quad \xi_{(0)}^1 = \xi_{(1)}^1 = \eta_1^{(0)} = \eta_1^{(1)} = 0. \tag{33}$$

The generators (33) correspond to the form invariance of the system.

Obviously, if we take G=0, then the generators (33) satisfy the generalized Noether identity (14), and therefore they correspond to the Noether symmetry of the system.

It is easy to prove that the generators (33) satisfy the conditions (24) and (25), therefore, according to Theorem 4, the generators (33) also correspond to the Lie symmetry of the system.

From Theorem 5, we obtain a conserved quantity of form invariance of the system as follows:

$$I = p_1^{(0)} q_{(1)}^1 + \frac{1}{2\alpha_2}(p_1^{(1)})^2 - \frac{1}{2}\alpha_1 (q_{(1)}^1)^2 = \text{constant}. \tag{34}$$

Here the expression (34) is also a conserved quantity of Noether symmetry of the system.

References

[1] Noether A E 1918 *Nachr. kgl Ges. Wiss. Göttingen. Math. Phys.* KI **II** 235

[2] Li Z P 1993 *Classical and Quantal Dynamics of Constrained Systems and their Symmetrical Properties* (Beijing: Beijing Polytechnic University Press) p1 (in Chinese)

[3] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press) p1 (in Chinese)

[4] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) p1 (in Chinese)

[5] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973

[6] Hojman S A 1992 *J. Phys. A: Math. Gen.* **25** L291

[7] Mei F X, Shang M 2000 *Acta Phys. Sin.* **49** 1901 (in Chinese)

[8] Zhang Y, Shang M and Mei F X 2000 *Chin. Phys.* **9** 401

[9] Mei F X 2000 *J. Beijing Inst. Technol.* **9** 120

[10] Mei F X 2001 *Chin. Phys.* **10** 177

[11] Mei F X 2001 *J. Beijing Inst. Technol.* **21** 535 (in Chinese)

[12] Wang S Y and Mei F X 2001 *Chin. Phys.* **10** 373

[13] Wang S Y and Mei F X 2002 *Chin. Phys.* **11** 5

[14] Luo S K 2002 *Chin. Phys. Lett.* **19** 449

[15] Ge W K 2002 *Acta Phys. Sin.* **51** 939 (in Chinese)