Vol 12 No 10, October 2003
1009-1963/2003/12(10)/1049-05

Chinese Physics

© 2003 Chin. Phys. Soc.
and IOP Publishing Ltd

Evolution property of soliton solutions for the Whitham–Broer–Kaup equation and variant Boussinesq equation*

Lin Ji(林 机)[a)b)†], Xu You-Sheng(许友生)[a)], and Wu Feng-Min(吴锋民)[a)]

[a)] *Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua* 321004, *China*

[b)] *Department of Physics, Shanghai Jiaotong University, Shanghai* 200030, *China*

(Received 17 February 2003; revised manuscript received 21 April 2003)

Using the standard Painlevé analysis approach, the (1+1)-dimensional Whitham–Broer–Kaup (WBK) and variant Boussinesq equations are solved. Some significant and exact solutions are given. We investigate the behaviour of the interactions between the multi-soliton-kink-type solution for the WBK equation and the multi-solitonic solutions and find the interactions are not elastic. The fission of solutions for the WBK equation and the fusions of those for the variant Boussinesq equation may occur after their interactions.

Keywords: soliton solution, soliton fission and fusion, Painlevé analysis approach, (1+1)-dimensional partial differential equation

PACC: 0200, 0420J, 1110L

1. Introduction

To search for exact solutions of nonlinear partial differential equations(PDEs) is one of the most prominent topics in the soliton theory, because of the wide applications of the soliton theory in mathematics, physics, chemistry, biology and communication, etc. There are various useful methods developed by mathematicians and physicists to find out special solutions of a PDE. Some of the most important methods are the inverse scattering transformation approach,[1] the bilinear method,[2] symmetry reduction,[3] Bäcklund and Darboux transformations,[4] mapping and deformation approach,[5] standard and extended truncated Painlevé analysis,[6] formal variable separation approach[7] and the nonformal variable separation method,[8], etc.

Usually, the interactions between soliton solutions obtained by the methods mentioned above for the integrable models are considered to be completely elastic. That is to say, the amplitude, velocity and wave shape of a soliton do not change after nonlinear interaction. However, for some special solutions of some (2+1)-dimensional models, the interactions among solitonic excitations (such as compactons, peakons) are not completely elastic,[9] their shapes do change after interactions. For some (1 + 1)-dimensional models, two or more solitons may fuse into one soliton at a specific time, while sometimes one soliton may fission into two or more solitons at that specific time.[10] We call these phenomena soliton fusion and soliton fission respectively. It is important and significant in physics and mathematics to study the evolutions of soliton solutions for the PDE with time. In fact, the phenomenon of soliton fission and fusion have been observed in many real physical system (such as organic membrane and macromolecular material,[11] Sr-Ba-Ni oxidation crystal and waveguides,[12] even-clump DNA[13] and many physical fields like plasma physics, nuclear physics, hydrodynamics and so on[14]).

In this paper, we study the evolutions of the soliton solutions for two (1 + 1)-dimensional PDEs with time. We are devoted to the investigation of the soliton solutions of the (1 + 1)-dimensional Whitham–Broer–Kaup equation and the soliton fissions in Section 2. In Section 3, we give the evolutions of the soliton solutions of the variant Boussinesq equation. A short summary and discussion is given in Section 4.

*Project supported by the National Outstanding Youth Foundation of China (Grant No 19925522), the National Natural Science Foundation of China (Grant No 90203001) and the Natural Science Foundation of Zhejiang Province, China (Grant No 102053).

†Corresponding author. E-mail: jhlinji@mail.jhptt.zj.cn

2. The soliton solutions and their fissions for the WBK equation

The WBK equation is a complete integrable model which describes the dispersive long-wave in shallow water[15]

$$u_t + uu_x + v_x + au_{xx} = 0, \tag{1}$$
$$v_t + vu_x + uv_x + bu_{xxx} - av_{xx} = 0, \tag{2}$$

where a and b are dispersive constants. With the help of the standard order leading analysis, the truncated Painlevé expansion of the WBK equation can be written as

$$u = \frac{u_0}{f} + u_1, \quad v = \frac{v_0}{f^2} + \frac{v_1}{f} + v_2, \tag{3}$$

where $\{u_1,\ v_2\}$ is the seed solution of the WBK equation. In order to gain some new solutions, we let the seed solution $u_1 = 0,\ v_2 = 0$ and substitute Eq.(3) into Eqs.(1) and (2), then we have

$$\begin{aligned}&(u_{0t} + au_{0xx} + v_{1x})f^{-1} + (v_{0x} - u_0 f_t + u_0 u_{0x}\\ &- au_0 f_{xx} - v_1 f_x - 2af_x u_{0x})f^{-2}\\ &+ (-u_0^2 f_x - 2v_0 f_x + 2au_0 f_x^2)f^{-3} = 0,\end{aligned} \tag{4}$$

$$\begin{aligned}&(v_{1t} + bu_{0xxx} - av_{1xx})f^{-1} + (u_0 v_{1x} - av_{0xx} + v_{0t}\\ &+ 2av_{1x}f_x + v_1 u_{0x} - v_1 f_t + av_1 f_{xx}\\ &- 3bu_{0x}f_{xx} - bu_0 f_{xxx} - 3bf_x u_{0xx})f^{-2}\\ &+ (6bu_0 f_x f_{xx} - 2f_t v_0 + 6bf_x^2 u_{0x} + v_0 u_{0x} + u_0 v_{0x}\\ &- 2v_1 u_0 f_x + 2av_0 f_{xx} - 2av_1 f_x^2 + 4av_{ox}f_x)f^{-3}\\ &+ (-6bu_0 f_x^3 - 3u_0 v_0 f_x - 6av_0 f_x^2)f^{-4} = 0.\end{aligned} \tag{5}$$

In order to obtain u_0, v_0 and v_1, we let the coefficient of f^{-2} in Eq.(4) and those of f^{-4}, f^{-3} in Eq.(5) vanish and obtain

$$\begin{aligned}u_0 &= 2\sqrt{b+a^2}f_x,\\ v_0 &= 2(a\sqrt{b+a^2} - b - a^2)f_x^2,\end{aligned} \tag{6}$$

and

$$\begin{aligned}v_1 =& -2\frac{(a\sqrt{b+a^2} - b - a^2)f_t}{2\sqrt{b+a^2}+a}\\ &-2\frac{\sqrt{b+a^2}(-3b-2a^2) + 2a(a^2+b)}{2\sqrt{b+a^2}+a}f_x^2.\end{aligned} \tag{7}$$

Substituting Eqs.(6) and (7) into Eqs.(4) and (5), we can obtain the same equation — the heat conduction equation from Eqs.(4) and (5)

$$f_t = -\sqrt{b+a^2}f_{xx}. \tag{8}$$

If we take the solutions of the heat conduction equation(8) as

$$\begin{aligned}f =& A + \sum_{i=1}^{N} a_i \exp(k_i x - \sqrt{a^2+b}k_i t + x_{i0})\\ &+ \sum_{i=1}^{N_1} b_i \cosh(q_i x - 2\sqrt{b+a^2}q_i Q_i t + c_{i0})\\ &\times \exp(Q_i x - \sqrt{b+a^2}(q_i^2 + Q_i^2)t + C_{i0}),\end{aligned} \tag{9}$$

the multi-soliton solutions of the WBK equation are given from Eqs.(6), (7), (9) and (3). In order to investigate the evolution of the multi-soliton solution with time, we take the two soliton-kink solutions as an example. We select Eq.(9) as follows:

$$\begin{aligned}f =& A + \cosh(q_1 x - 2\sqrt{b+a^2}q_1 Q_1 t)\\ &\times \exp(Q_1 x - \sqrt{b+a^2}(q_1^2 + Q_1^2)t),\end{aligned} \tag{10}$$

and let

$$a = 1, \quad b = 3, \quad A = 3, \quad q_1 = 1, \quad Q_1 = -1/3, \tag{11}$$

then we can obtain the two soliton-kink solutions of the WBK equation

$$u = -4\frac{\exp\Big(-\frac{1}{3}x - \frac{20}{9}t\Big)\Big(\sinh\Big(x+\frac{4}{3}t\Big) - \frac{1}{3}\cosh\Big(x+\frac{4}{3}t\Big)\Big)}{\Big(3+\cosh\Big(x+\frac{4}{3}t\Big)\exp\Big(-\frac{1}{3}x - \frac{20}{9}t\Big)\Big)}, \tag{12}$$

$$\begin{aligned}v =& -2\frac{\Big(\frac{10}{9}\cosh\Big(x+\frac{4}{3}t\Big)^2 - \frac{2}{3}\sinh\Big(x+\frac{4}{3}t\Big)\cosh\Big(x+\frac{4}{3}t\Big) - 3\Big)\exp\Big(-\frac{2}{3}x - \frac{40}{9}t\Big)}{\Big(3+\cosh\Big(x+\frac{4}{3}t\Big)\exp\Big(-\frac{1}{3}x - \frac{20}{9}t\Big)\Big)^2}\\ &-2\frac{4\sinh\Big(x+\frac{4}{3}t\Big) - \frac{20}{3}\cosh\Big(x+\frac{4}{3}t\Big)}{\Big(3+\cosh\Big(x+\frac{4}{3}t\Big)\exp\Big(-\frac{1}{3}x - \frac{20}{9}t\Big)\Big)^2}\exp\Big(-\frac{1}{3}x - \frac{20}{9}t\Big).\end{aligned} \tag{13}$$

Figure 1 shows the evolutions of the soliton-kink solution for the field v (Eq.(13)) and the kink solution for the field u (Eq.(12)) with time. From Fig.1, we can see that one soliton-kink solution for v (or one kink solution for u) fission to two soliton-kink solution (or two kink-type solutions). By careful analysis, we find that only soliton fission occurs for the fields v and u for all the ranges of the two arbitrary parameters q_1 and Q_1 in Eq.(10). Furthermore, we study the function f of the general form Eq.(9) for one case of $N_1 = 0$, $N = 2$ and for another case of $N = 2$, $N_1 = 2$. The same results are given. In other words, neither elastic scattering nor soliton fusion exists for the WBK system.

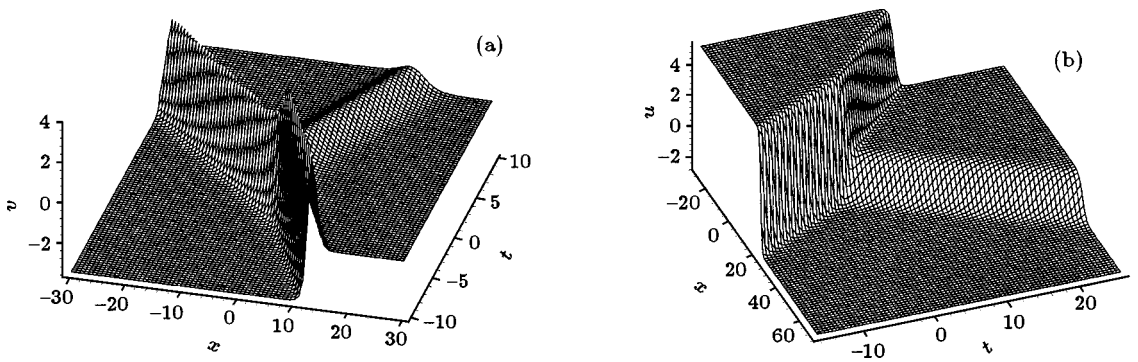


Fig.1. Plots of fissions for the soliton-kink solutions of the WBK equation: (a) v, (b) u.

3. The soliton solutions and their fusions for the variant Boussinesq equation

The variant Boussinesq equation

$$u_t + v_x + uu_x = 0, \tag{14}$$

$$v_t + (uv)_x + u_{xxx} = 0 \tag{15}$$

has been studied for its integrability and solutions.[16] Similar to the previous section, using the standard truncated Painlevé extension method, we have the Bäcklund transformation as follows:

$$u = -2\frac{f_x}{f} + u_0, \quad v = -2\frac{f_x^2}{f^2} + 2\frac{f_t}{f} + v_0, \tag{16}$$

where f is the function of two variables x and t, and $\{u_0,\ v_0\}$ is a seed solution of Eqs.(14) and (15). In order to obtain more solutions of Eqs.(14) and (15), we discuss the two cases. Firstly, we take the seed solution to be zero (i.e. $u_0 = v_0 = 0$) and substitute Eq.(16) into Eqs.(14) and (15), then we obtain the same relation for the function f just as the WBK equation from the variant Boussinesq equation

$$f_t - f_{xx} = 0. \tag{17}$$

So f has the solution like Eq.(9). We obtain multi-soliton-type solutions for the fields u and v. In Fig.2, we give the four soliton-type solutions of the fields u and v which are fused into one-soliton-type solution at time $t = 0$ for the function of f in the following form:

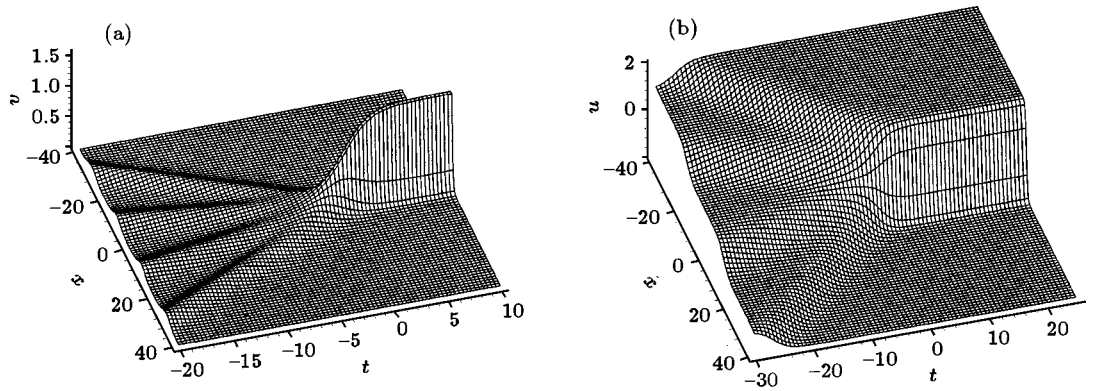


Fig.2. Fusions of the four solitons for the variant Boussinesq equation with Eq.(18) and $k_1 = 1/2$, $k_2 = 1$, $k_3 = -1, k_4 = -1/2$: (a) v, (b) u.

$$f = 1 + \exp(k_1 x + k_1^2 t) + \exp(k_2 x + k_2^2 t) + \exp(k_3 x + k_3^2 t) + \exp(k_4 x + k_4^2 t). \quad (18)$$

Secondly, we take the seed solution not equal to zero and substitute Eq.(16) into Eqs.(14) and (15), then we have

$$v_{0x} + u_0 u_{0x} + u_{0t} - 2(f_x u_{0x} - f_{xxx} + f_{xt} + u_0 f_{xx}) f^{-1} + 2(u_0 f_x^2 - f_{xx} f_x + f_t f_x) f^{-2} = 0, \quad (19)$$

$$u_{0x} v_0 + v_{0t} + u_0 v_{0x} + u_{0xxx} + 2(u_0 f_{xxx} - f_{xxxx} - f_x v_{0x} - f_{xx} v_0 + u_{0x} f_{xx} + f_{xxt}) f^{-1} + (4 f_{xxx} f_x - 2 f_{xx} f_t - 6 u_0 f_{xx} f_x + 2 f_x^2 v_0 + 2 f_{xx}^2 - 2 u_{0x} f_x^2 - 4 f_x f_{xt}) f^{-2} + 4(-f_x^2 f_{xx} + f_t f_x^2 + u_0 f_x^3) f^{-3} = 0. \quad (20)$$

We are able to solve the seed solution u_0, v_0 by vanishing the coefficient of f^{-2} in Eq.(19) (or f^{-3} in Eq.(20)) and that of f^{-2} in Eq.(20), and obtain

$$u_0 = \frac{f_{xx} - f_t}{f_x},$$
$$v_0 = -\frac{f_{xxx} f_x - f_{xt} f_x + f_{xx} f_t - f_{xx}^2}{f_x^2}, \quad (21)$$

and that the function of f satisfies the following equations:

$$(2 f_{xxt} - f_{xxxx} - f_{tt}) f_x^2 + 2(2 f_{xxx} f_{xx} + f_{xt} f_t - f_{xxx} f_t - 2 f_{xx} f_{xt}) f_x - f_{xx} f_t^2 + 4 f_{xx}^2 f_t - 3 f_{xx}^3 = 0, \quad (22)$$

and

$$(f_{xxxxx} - 2 f_{xxxt} + f_{xtt}) f_x^3 + (-2 f_{xxt} f_t - 2 f_{xt}^2 + 6 f_{xx} f_{xxt} - 4 f_{xxx}^2 + 6 f_{xxx} f_{xt} + 2 f_{xxxx} f_t - 5 f_{xx} f_{xxxx} - f_{xx} f_{tt}) f_x^2 + (f_{xxx} f_t^2 + 17 f_{xxx} f_{xx}^2 - 12 f_{xt} f_{xx}^2 + 6 f_{xt} f_{xx} f_t - 12 f_{xxx} f_{xx} f_t) f_x - 3 f_{xx}^2 f_t^2 + 12 f_{xx}^3 f_t - 9 f_{xx}^4 = 0. \quad (23)$$

It is difficult that the general solution of the function f satisfies the trilinear Eq.(22) and the quatrilinear Eq.(23) is obtained. However, If we take the following form of the function f

$$f = 1 + \sum_{i=1}^{N} \exp(k_i x + \omega_i t + x_{0i}), \quad (24)$$

where k_i and ω_i satisfy the following dispersion relation:

$$\omega_i = \frac{k_i (k_1 k_i - k_1^2 + \omega_1)}{k_1}, \quad (25)$$

then we can obtain the multi-soliton solution of the system expressed by Eqs.(14) and (15). It is understood from Eq.(25) that there is not any dispersion relation for the single soliton solution ($N = 1$ for Eq.(24)). From Fig.3, we know that the interaction of the multi-soliton solution is not elastic because the three-soliton solution can finally be fused to one soliton solution after a special time.

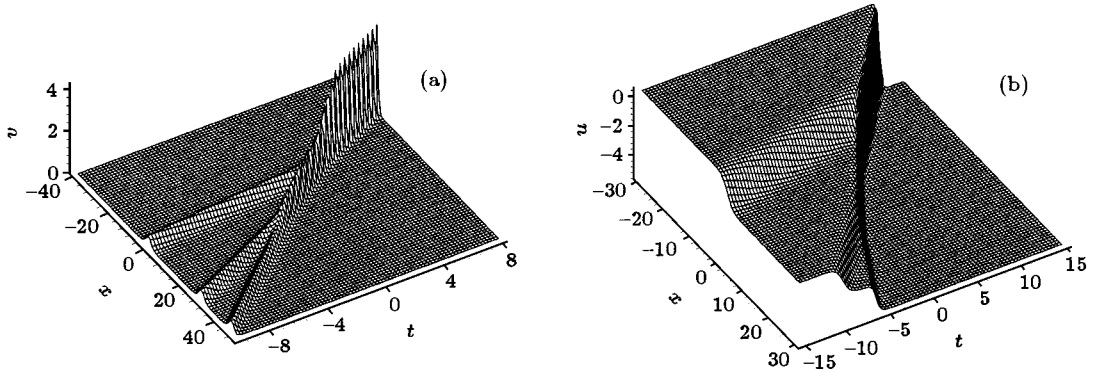


Fig.3. Evolutions of the three-soliton fusions for the variant Boussinesq equation with Eq.(24) and $N = 3$, $k_1 = 2$, $k_2 = 3$, $k_3 = 1$, $\omega_1 = 3$, $x_{10} = 0$, $x_{20} = -2$, $x_{30} = 5$: (a) v, (b) u.

4. Summary and discussion

We have solved the WBK equation and the variant Boussinesq equation by the standard Painlevé method and obtained multi-soliton solutions and also investigated the interactions among solitons. We find

that the interactions among solitons are not elastic. On the one hand, multi-solitons can be fused finally to one soliton at a special time. On the other hand, a single-soliton can also fission into multi-solitons. Generally, the interaction among solitons is completely elastic for a lot of integrable models, namely, the soliton's amplitudes, velocities and/or wave shapes will not change after nonlinear interactions. Why do these phenomena occur? Comparing the N fusion(or fission) soliton solution (such as Eq.(24)) and the N-soliton solution with Hirota's traditional elastic interaction, like N-soliton expression of KdV-type equation($\eta_i = k_i x + \omega_i t$)[2]

$$\begin{aligned} f =& 1 + \sum_{i=1}^{N} \exp(\eta_i) + \sum_{i \le j} A_{ij} \exp(\eta_i + \eta_j) \\ &+ \sum_{i \le j \le k} A_{ij} A_{ik} A_{jk} \exp(\eta_i + \eta_j + \eta_k) \\ &+ L + \prod_{i \le j} A_{ij} \exp\left(\sum_{k=1}^{N} \eta_k \right). \end{aligned} \tag{26}$$

We find that the difference lies in that the coupling constants A_{ij} of Eq.(26) are not zero. This indicates that soliton fission and fusion cannot occur in those models in which coupling constants cannot be fixed to zero by different selections of the parameters(k_i, ω_i). In contrast, they exist in some models such as the variant Boussinesq equation, the WBK equation, Boussinesq equation, the Kupershmidt equation and (2+1)-dimensional KdV equation.[10] In fact, if a coupling constant $A_{i_1 j_1}$ is zero while other A_{ij} are non-zero, then the i_1th soliton and the j_1th soliton must fission or fusion at some time. Furthermore, the fission or fusion will not occur between other solitons. Hence,

$$A_{i_1 j_1} = 0$$

may be the sufficient and necessary condition to determine whether or not the i_1th and j_1th solitons can fission or fuse.

Acknowledgements

One of the authors (Lin) would like to thank Professor S-Y Lou and Drs. Tang X Y, Zhang S L, Qian X M, Hu H C, Chen C L and Ma H C for useful discussions.

References

[1] Gardner C S, Green J M, Kruskal M D and Miura R M 1967 *Phys. Rev. Lett.* **19** 1095

[2] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192

[3] Olver P J 1986 *Application of Lie Groups to Differential Equation* (Berlin: Springer)
Clarkson P A and Kruskal M D 1989 *J. Math. Phys.* **30** 2201
Lou S Y and Tang X Y 2001 *Chin. Phys.* **10** 891
Lou S Y, Ruan H Y, Chen D F and Chen W Z 1991 *J. Phys. A: Math. Gen.* **24** 1455

[4] Gu C H 1992 *Lett. Math. Phys.* **26** 199

[5] Lou S Y 1999 *J. Phys. A: Math. Gen.* **32** 4521
LI H M 2002 *Chin. Phys.* **11** 1111

[6] Lou S Y 1998 *Z. Naturf.* **53a** 251

[7] Lou S Y and Chen L L 1999 *J. Math. Phys.* **40** 6491

[8] Tang X Y, Lou S Y and Zhang Y 2002 *Phys. Rev.* E **66** 046601
Lin J and Li H M 2002 *Z. Naturf. A: Phys. Sci.* **57** 929

[9] Lou S Y 2002 *J. Phys. A: Math. Gen.* **35** 10619
Lin J and Qian X M 2003 *Phys. Lett.* A **313** 93

[10] Ying J P 2001 *Commun. Theor. Phys.* **35** 405

[11] Serkin V N, Chapela V M, Percino J and Belyaeva T L 2001 *Opt. Commun.* **192** 237

[12] Krolikowski W and Holmstrom S A 1997 *Opt. Lett.* **22** 15
Kip D, Wesner M, Herden C and Shandarov V 1999 *Appl. Phys. B: Lasers Opt.* **68** 971

[13] Hisakado M 1997 *Phys. Lett.* A **227** 87

[14] Ono H and Nakata I 1994 *J. Phys. Soc. Jpn.* **63** 40
Stoitcheva G, Ludu L and Draayer J P 2001 *Math. Computers Simulations* **55** 621

[15] Kupershmidt B A 1985 *Commun. Math. Phys.* **99** 51

[16] Wang M L 1995 *Phys. Lett.* A **199** 279