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New amplitude equation of single-mode laser*

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The white-gain model and the white-loss model of a single-mode laser are investigated in the presence of cross-correlations between the real and imaginary parts of quantum noise as well as pump noise. It was found that, like the white cubic model (2001 *Chin. Phys. Lett.* **18** 370), the amplitude equations are all decoupled from the phase equations for the two models, and the same novel term appears in the amplitude equations of the two models. So we can put the amplitude equations of all the models into a general form, that is, the new amplitude equation. We further use this new amplitude equation to study specifically the stationary properties of the laser intensity for the white-gain model.

Keywords: single-mode laser, phase lock, new equation of amplitude, white-gain model
PACC: 4250, 0540, 4260K

1. Introduction

From the late 1980s to the early 1990s the statistical properties of a single-mode laser with quantum noise and pump noise had been widely studied. But the quantum noise and the pump noise in the laser field were usually taken as uncorrelated.[1−3] With the appearance of new phenomenon in the laser field, people began to explore the new statistical properties by taking into account the correlation between the quantum noise and the pump noise.[4−7] Zhou *et al* assumed the cross-correlation between the real and imaginary parts of the quantum noise,[8] but in their specific calculation the amplitude equation was not decoupled from the phase equation, and they dealt with it in the two-dimensional case. Soon afterwards Ke *et al*[9] also considered the cross-correlation between the real and imaginary parts of quantum noise, but they adopted a new method to make the amplitude equation decoupled from the phase equation, whereby they derived the amplitude equation of the cubic model for single-mode laser, in which a novel term $\frac{Q(1-|\lambda_q|)}{2r}$ appeared. Fox and Roy have already proved that the laser equation has a novel term $\frac{R}{2y}$,[10] and pointed out that the amplitude equation without this term was uncorrected. However, the amplitude equation of Ref.[9] not only contained a novel term, but also the term can be changed by adjusting the value of the cross-correlations coefficient $\lambda_q(0 \leq |\lambda_q| \leq 1)$ between the real and imaginary parts of the quantum noise.

In the present paper, based on the investigation of Ref.[9], we extend the study to two more accurate models. We find that the amplitude equations of both models can be decoupled from the phase equations of the models, and the term in amplitude equations is the same as that in the cubic model. What is more, the cross-correlation coefficient λ_p between the real and imaginary parts of the pump noise is not contained in the amplitude equation, therefore we can write the amplitude equations of all kinds of models in a general form, namely, the new equation obtained in section II. In section III we calculate specifically the stationary intensity distribution (SID) of white-gain model on the basis of the new amplitude equation, and the result shows that there exists a first-order-like phase transition. Then the mean, normalized variance and skewness of the laser intensity are calculated, and some new phenomena appear, which the cubic model does not have. Section IV are our conclusions and discussions.

2. The new amplitude equation

We consider two more accurate laser models, which do not ignore the high order effect, namely the white-gain model and white-loss model. They

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can be described by Langevin equations with cross-correlation between the real and imaginary parts of the quantum noise as well as the pump noise

$$\dot{E} = \left(\frac{A_{\rm g}}{1+|E|^2} - a_{\rm l}\right)E + \frac{E}{1+|E|^2}p(t) + q(t), \quad (1)$$

$$\dot{E} = -kE + \frac{\Gamma E}{1+\beta|E|^2} + Ep(t) + q(t). \quad (2)$$

Electric field E in Eqs.(1) and (2) is the complex laser field, real parameters $A_{\rm g}$ and $a_{\rm l}$ in Eq.(1) denote gain and loss coefficients, real parameters k and Γ in Eq.(2) are loss and gain coefficient, β is a positive parameter.

The random variable $p(t)$ and $q(t)$ represent the complex pump and quantum noise, the noise terms are assumed to be Gaussian white noises with zero mean and correlations

$$\begin{aligned} &< q_i(t)q_j(s) > \\ =&[\delta_{ij} + \lambda_q(1-\delta_{ij})]Q\delta(t-s) \qquad (i,j = I,R), \\ &< p_i(t)p_j(s) > \\ =&[\delta_{ij} + \lambda_p(1-\delta_{ij})]Q'\delta(t-s) \qquad (i,j = I,R), \\ &< q_i(t)p_j(s) >= 0 \qquad (i,j = I,R), \end{aligned} \quad (3)$$

where I and R denote the real and imaginary part, respectively. λ_p and λ_q are the cross-correlation coefficients, which give a measure of the degree of cross-correlation between the real and imaginary parts of the pump noise as well as quantum noise, respectively, and $|\lambda_p| \le 1, |\lambda_q| \le 1$. Q and Q' are the quantum and pump-noise strengths.

Writing Eqs. (1) and (2) in polar coordinates, $E = re^{{\rm i}\phi}$, r being the amplitude of the laser field, ϕ the phase of laser field, Eq.(1) then reads

$$\frac{{\rm d}r}{{\rm d}t} = \left(\frac{A_{\rm g}}{1+r^2} - a_{\rm l}\right)r + \frac{p_{\rm R}(t)r}{1+r^2} + \varepsilon_r(t), \quad (4)$$

$$\frac{{\rm d}\phi}{{\rm d}t} = \frac{p_{\rm I}(t)}{1+r^2} + \frac{\varepsilon_\phi(t)}{r}, \quad (5)$$

where

$$\varepsilon_r(t) = q_{\rm R}(t)\cos\phi + q_{\rm I}(t)\sin\phi, \quad (6)$$

$$\varepsilon_\phi(t) = -q_{\rm R}(t)\sin\phi + q_{\rm I}(t)\cos\phi. \quad (7)$$

Since $< \varepsilon_r(t) > \neq 0, < \varepsilon_\phi(t) > \neq 0$, applying the techniques developed in Ref.[1], $\varepsilon_r(t)$ and $\varepsilon_\phi(t)$ can be changed to

$$\varepsilon_r(t) = \frac{Q}{2r}(1 - \lambda_q\sin 2\phi) + \tilde{\varepsilon}_r(t), \quad (8)$$

$$\varepsilon_\phi(t) = -\frac{Q}{2r}\lambda_q\cos 2\phi + \tilde{\varepsilon}_\phi(t), \quad (9)$$

where pure noise $\tilde{\varepsilon}_r(t)$ and $\tilde{\varepsilon}_\phi(t)$ are zero mean with correlations

$$\begin{aligned} &< \tilde{\varepsilon}_r(t)\tilde{\varepsilon}_r(s) >= Q(1+\lambda_q\sin 2\phi)\delta(t-s), \\ &< \tilde{\varepsilon}_\phi(t)\tilde{\varepsilon}_\phi(s) >= Q(1-\lambda_q\sin 2\phi)\delta(t-s), \\ &< \tilde{\varepsilon}_r(t)\tilde{\varepsilon}_\phi(s) >= Q\lambda_q\cos 2\phi\delta(t-s). \end{aligned} \quad (10)$$

Inserting Eqs.(8) and (9) into Eqs.(4) and (5), respectively, we have

$$\begin{aligned} \frac{{\rm d}r}{{\rm d}t} =& \left(\frac{A_{\rm g}}{1+r^2} - a_{\rm l}\right)r + \frac{p_{\rm R}(t)r}{1+r^2} \\ &+ \frac{Q}{2r}(1-\lambda_q\sin 2\phi) + \tilde{\varepsilon}_r(t), \end{aligned} \quad (11)$$

$$\frac{{\rm d}\phi}{{\rm d}t} = \frac{p_{\rm I}(t)}{1+r^2} - \frac{Q}{2r^2}\lambda_q\cos 2\phi + \frac{\tilde{\varepsilon}_\phi(t)}{r}. \quad (12)$$

From Eq.(12) we can see clearly that the presence of a new drift term, $-(Q/2r^2)\lambda_q\cos 2\phi$, arising from the nonzero cross-correlation coefficient λ_q, leads the laser phase to be locked. Equation (12) is then given by

$$\frac{{\rm d}\phi}{{\rm d}t} = -\frac{Q}{2r^2}\lambda_q\cos 2\phi + f(t), \quad (13)$$

and the statistical properties of $f(t)$ is

$$< f(t)f(s) >= \left[\frac{Q'}{(1+r^2)^2} + \frac{Q(1-\lambda_q\sin 2\phi)}{r^2}\right]\delta(t-s). \quad (14)$$

The condition of phase lock is: $< {\rm d}\phi/{\rm d}t >= 0$, the most stable lock is subject to

$$\frac{\partial({\rm d}\phi/{\rm d}t)}{\partial\phi} - \frac{1}{2}\frac{\partial^2 < f^2(t) >}{\partial\phi^2} < 0. \quad (15)$$

Both the condition of phase lock and the term of the most stable lock determine that the phase will be locked at a stable value $\phi_{\rm s}$, satisfying the following equation

$$\lambda_q\sin 2\phi = |\lambda_q|. \quad (16)$$

From Eq.(16), we have

$$\phi_{\rm s} = (2n+1)\pi/4, \qquad n = 0, \pm 1, \pm 2, \cdots$$

Inserting Eq.(16) into Eq.(11), we have

$$\begin{aligned} \frac{{\rm d}r}{{\rm d}t} =& \left(\frac{A_{\rm g}}{1+r^2} - a_{\rm l}\right)r + \frac{p_{\rm R}(t)r}{1+r^2} \\ &+ \frac{Q}{2r}(1-|\lambda_q|) + \tilde{\varepsilon}_r(t). \end{aligned} \quad (17)$$

In the same way, we can obtain the white-loss model, namely, Eq.(2) also meet the phase lock Eq. (16), the

phase will be locked, the amplitude equation is decoupled from the phase equation, too. The corresponding amplitude equation is

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -kr + \frac{\Gamma r}{1+\beta r^2} + rp_{\mathrm{R}}(t) + \frac{Q(1-|\lambda_q|)}{2r} + \tilde{\varepsilon}_r(t), \tag{18}$$

the value of the phase lock ϕ_{s} is the same as that of the white-gain model.

The amplitude equation of white cubic model in Ref.[9] is cited as follows

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t} =& a_0 r - Ar^3 + \frac{\Gamma r}{1+\beta r^2} + rp_{\mathrm{R}}(t) \\ &+ \frac{p(1-|\lambda_q|)}{2r} + \tilde{\varepsilon}_r(t).\end{aligned} \tag{19}$$

Depending on the deriving of the amplitude equations for the above two models and the conclusion of cubic model in Ref.[9], we can draw a conclusion: if the noise has the characteristic of Eq.(3), then whatever models we choose, they all fulfil the phase lock equation, the value of phase lock is the same, the amplitude equations possess the same new term. So we can have the amplitude equations of all kinds of laser models written in a general form, that is the new amplitude equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = f(r) + g(r)p_{\mathrm{R}}(t) + \frac{Q(1-|\lambda_q|)}{2r} + \tilde{\varepsilon}_r(t), \tag{20}$$

where

$$\begin{aligned}&<\tilde{\varepsilon}_r(t)>=0, \\ &<\tilde{\varepsilon}_r(t)\tilde{\varepsilon}_r(s)>=Q(1+|\lambda_q|)\delta(t-s).\end{aligned} \tag{21}$$

In what follows, we investigate specifically the statistical properties of the laser intensity for white-gain model on the basis of the above analysis, and some new phenomenon is obtained.

3. The SID and the moment of laser intensity for white-gain model

Combining Eq.(17) and $r=\sqrt{I}$, we can obtain the Langevin equation for the laser intensity

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} =& 2\left(\frac{A_{\mathrm{g}}}{1+I} - a_{\mathrm{l}}\right)I + \frac{2p_{\mathrm{R}}(t)I}{1+I} \\ &+ p(1-|\lambda_q|) + 2\sqrt{I}\tilde{\varepsilon}_r(t).\end{aligned} \tag{22}$$

The Fokker–Planck equation of the intensity probability distribution $Q(I,t)$ corresponding to Eq.(22) is given by

$$\begin{aligned}\frac{\partial Q(I,t)}{\partial t} =& -\frac{\partial}{\partial I}\left[\left(2\left(\frac{A_{\mathrm{g}}}{1+I} - a_{\mathrm{l}}\right)I + \frac{8I}{(1+I)^3}Q' + 5Q + 3Q|\lambda_q|\right)Q(I,t)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial I^2}\left[\left(\frac{4I^2}{(1+I)^2}Q' + 4IQ(1+|\lambda_q|)\right)Q(I,t)\right].\end{aligned} \tag{23}$$

The SID $Q(I,t)$ is obtained, and reads

$$\begin{aligned}Q(I) =& \frac{N_0}{\sqrt{\dfrac{I^2Q'}{(1+I)^2} + IM}} I^{\frac{W}{2M}}[MI^2 + (2M+Q')I + M]^{\frac{a_{\mathrm{l}}Q' + MA_{\mathrm{g}}}{2M^2}} \exp\left[-\frac{a_{\mathrm{l}}}{M}(1+I)\right. \\ &\left. + \frac{8MQ'a_{\mathrm{l}} + 4Q'^2a_{\mathrm{l}} + 4MA_{\mathrm{g}}Q' - 2WM^2 + 3WMQ'}{4M^2\sqrt{Q'^2+4MQ'}}\arctan\left(\frac{2MI+2M+Q'}{\sqrt{Q'^2+4Q'M}}\right)\right],\end{aligned} \tag{24}$$

where $W=Q(1-|\lambda_q|)$, $M=Q(1+|\lambda_q|)$, N_0 is the normalization constant. Further λ_q appears as an absolute value $|\lambda_q|$, in order to simplify the calculation we assume λ_q not to be negative.

The mean $<I>$, normalized variance $\lambda_2(0)$ and skewness $\lambda_3(0)$ of the intensity I are defined, respectively, as

$$<I>=\int_0^\infty IQ(I)\mathrm{d}I, \tag{25}$$

$$\lambda_2(0) = \frac{<I^2>}{<I>^2} - 1, \tag{26}$$

$$\lambda_3(0) = \frac{<I^3>}{<I>^3} - 3\lambda_2(0) - 1. \tag{27}$$

4. Conclusions and discussions

Based on the above calculations, we will discuss the following five aspects.

4.1. λ_p, the cross-correlation coefficient between the real and the imaginary parts of the pump noise is not contained in the amplitude equation

In Eqs. (1) and (2), although we simultaneously take into account the cross-correlation between the real and imaginary parts of the quantum noise (denoted by λ_q) as well as the pump noise (denoted by λ_p), after being decoupled, the amplitude Eq. (20) only contains λ_q, λ_p is not included. Why? It is because that before writing Fokker–Planck equation, we have used the method of functional[1,11] to have the Langevin equation decoupled about phase, which leads to the absence of λ_p in the amplitude Eq. (20). This shows though the method of decoupling in this paper has the virtue of simplicity, it needs to be mended in order to reflect the effect of λ_p on the statistical properties of laser intensity.

4.2. The SID of white-gain model

Figure 1 shows the distribution function $Q_{st}(I)$ as a function of the intensity I, where λ_q is a parameter. It is seen that the curve of distribution function $Q_{st}(I)$ exhibits three different shapes for different value of λ_q: no-extremum ($\lambda_q = 1$), single-extremum ($\lambda_q = 0$), and two-extremum ($\lambda_q = 0.5$), this means the SID has a "first-order-like phase transition". In order to understand the situation of three cases fully, we discuss the extreme values of SID. Let $\partial Q_{st}(I)/\partial I = 0$, we obtain the extremum equation from Eq.(24),

$$\frac{\left(\frac{A_g}{2M}+\frac{a_1 Q'}{2M^2}\right)(2M+Q'+2MI)}{M+(2M+Q')I+MI^2} - \frac{\left(M-\frac{2Q'I^2}{(1+I)^3}+\frac{2Q'I}{(1+I)^2}\right)}{2MI+\frac{Q'I^2}{(1+I)^2}} + \frac{W}{2MI} - \frac{a_1}{M} + \frac{(8a_1MQ'+4A_gMQ'-2M^2W+3MQ'W)}{2M(4MQ'+Q'^2+(2M+Q'+2MI)^2)} = 0. \quad (28)$$

This is a high power function of the intensity I. If we define the parameters A_g, a_1 and Q' as fixed values, and calculate the number of the extrema, we can plot the regions of the extrema in the parameter plane (λ_q, Q), as shown in Fig.2. I represents the single-extremum region, and, in fact, it is a straight line with $\lambda_q = 0$. II and III represent the two-extremum region and no-extremum region, respectively; both being divided into two equal parts by the straight line λ_q=0. It also can be found that when the strength of the pump noise Q' increases (the dotted line), the two-extremum region (II) expands and no-extremum region (III) contracts. In order to testify the correctness of the parameter figures, we plot the curves of $Q_{st}(I)$-I in the three phase regions, respectively, as shown in Figs.3–5.

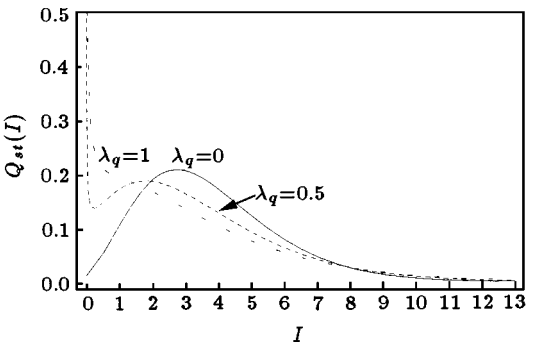


Fig.1. Stationary intensity distribution of the single-mode laser with Q=1, Q'=0.5, A_g=3, a_1=1; —— for λ_q=0, --- for λ_q=0.5 and − − − for λ_q=1.

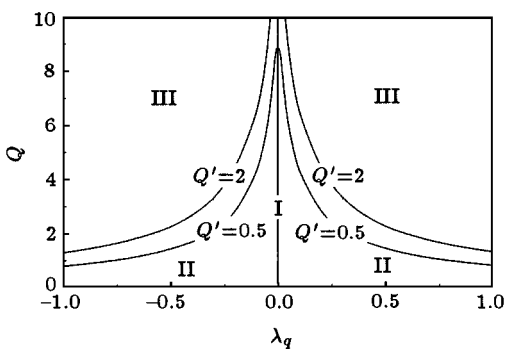


Fig.2. Parameter plane of the single-mode laser with A_g=3, a_1=1. Two critical curves are shown for Q'=0.5 and Q'=2. Region I indicated by the dotted line (λ_q=0) corresponds to the single-extremum structure of the stationary intensity distribution, and region II and region III correspond to the two-extremum and no-extremum structure, respectively.

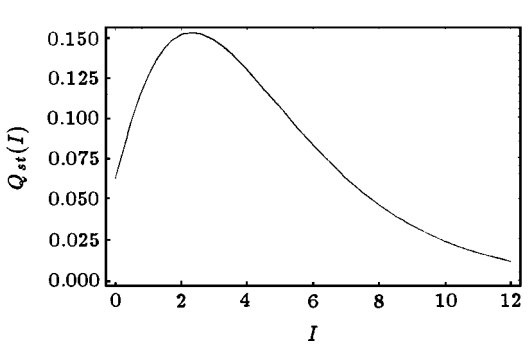


Fig.3. Stationary intensity distribution for Q=2, λ_q=0, with Q'=0.5, A_g=3 and a_1=1, corresponding to region I in Fig.2.

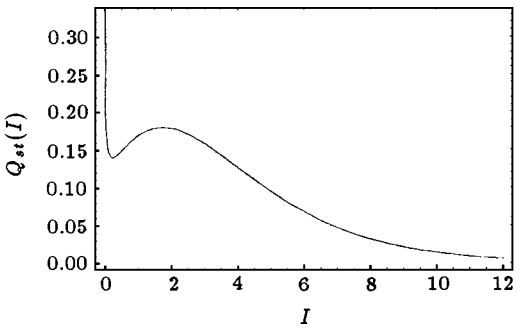


Fig.4. Stationary intensity distribution for Q=1.2, λ_q=0.5, with Q'=0.5, A_g=3 and a_1=1, corresponding to region II in Fig.2.

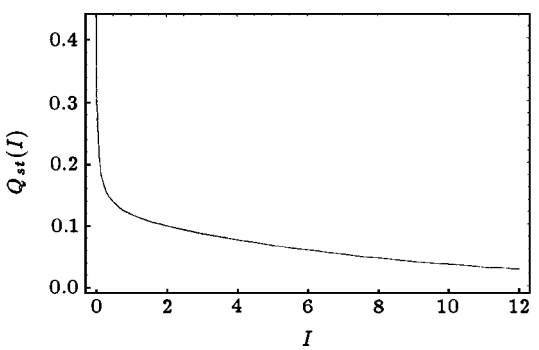


Fig.5. Stationary intensity distribution for Q=5, λ_q=0.5, with Q'=0.5, A_g=3 and a_1=1, corresponding to region III in Fig.2.

4.3. The mean laser intensity

We plot the mean laser intensity $< I >$ as a function of the pump parameter A_g in Figs.6 and 7. Figure 6 shows, when λ_q is a parameter and $Q < Q'$, the mean intensity $< I >$ increases linearly with A_g, the lines with different λ_q are parallel to each other, and the line moves down ward as λ_q increases. In Fig.7, when $Q > Q'$, the curves of $< I >$-A_g almost overlap at the beginning, without showing the effect of λ_q. With increasing A_g, the curves are slightly dependent on λ_q. Figure 8 shows the dependence of the mean laser intensity $< I >$ on a_1, indicating that $< I >$ monotonically decreases with a_1, and the parameters λ_q, Q and Q' have no obvious effect on the $< I >$-a_1 curves.

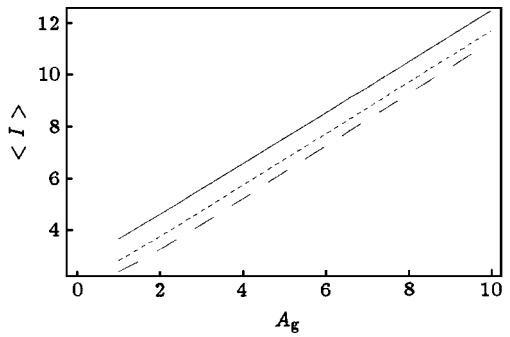


Fig.6. Mean steady-state laser intensity $< I >$ as a function of the pump parameter A_g with Q=1, Q'=2 and a_1=1; —— for λ_q=0, - - - for λ_q=0.5 and – – – for λ_q=1.

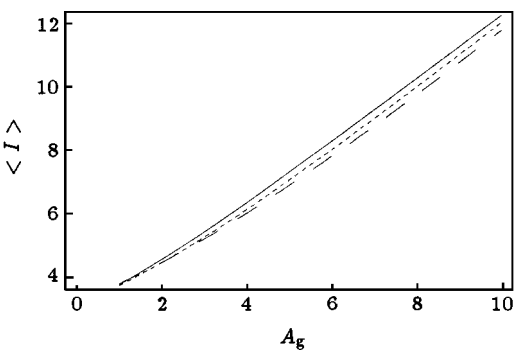


Fig.7. Mean steady-state laser intensity $< I >$ as a function of the pump parameter A_g with Q=3, Q'=0.5 and a_1=1; —— for λ_q=0, - - - for λ_q=0.5 and – – – for λ_q=1.

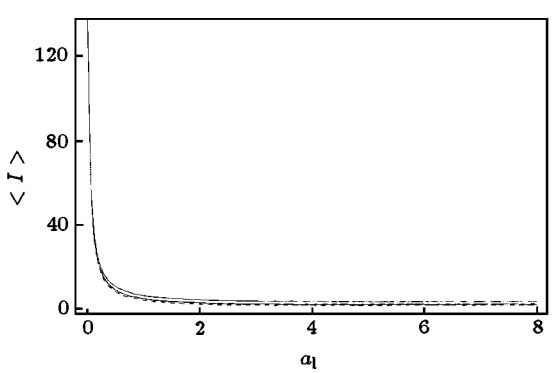


Fig.8. Mean steady-state laser intensity $< I >$ as a function of the loss coefficient a_1 with Q=1.5, Q'=1 and A_g=3; —— for λ_q=0, - - - for λ_q=0.5 and – – – for λ_q=1.

4.4. The normalized variance of the laser intensity

Figures 9 and 10 show the dependence of normalized variance $\lambda_2(0)$ on the pump parameter a_l for different values of Q'. It can be seen that the curves exhibit two shapes: one is the curve having a peak before decreasing to zero, and with decreasing Q' the magnitude of $\lambda_2(0)$ increases and the peak position shifts to the right. The other one is, when Q' increases, the curve decreases monotonically as a_l increases. Moreover, we can see for either $\lambda_q = 0$ or $\lambda_q \neq 0$, the curves of $\lambda_2(0)$-a_l always have the above two shapes. One may conclude that this phenomenon originates from the change of Q'. However, for the same Q', the magnitude of the correlated curve is greater than that of the uncorrelated, and the correlated curve changes more quickly than the uncorrelated. Figure 11 shows the $\lambda_2(0)$-A_g curves. It is clear that the normalize variance $\lambda_2(0)$ decreases monotonically with A_g, and the curve goes up as λ_q increases or Q' decreases.

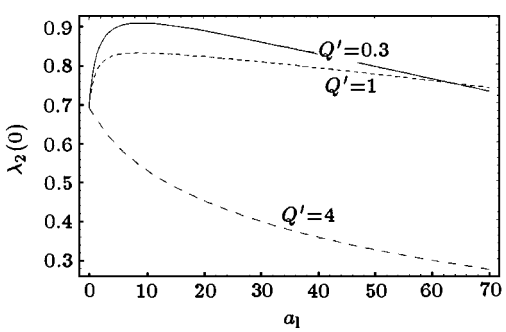


Fig.9. Normalized variance $\lambda_2(0)$ of the laser intensity as a function of the loss coefficient a_l. The parameters are Q=4.5, A_g=2 and λ_q=0; —— for Q'=0.3, - - - for Q'=1 and — — — for Q'=4.

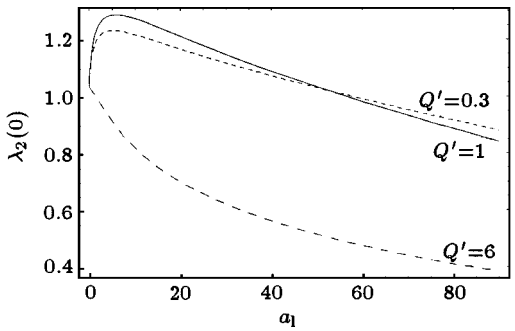


Fig.10. Normalized variance $\lambda_2(0)$ of the laser intensity as a function of the loss coefficient a_l. The parameters are Q=4.5, A_g=2 and λ_q=0.5; —— for Q'=0.3, - - - for Q'=1 and — — — for Q'=6.

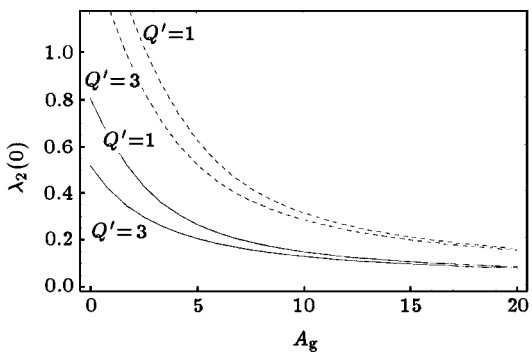


Fig.11. Normalized variance $\lambda_2(0)$ of the laser intensity as a function of the pump parameter A_g. The parameters are Q=1.5, a_l=1; —— for λ_q=0 with Q'=1 and Q'=3, - - - for λ_q=1 with Q'=1 and Q'=3.

4.5. The skewness $\lambda_3(0)$ of the laser intensity

The relationship of the skewness $\lambda_3(0)$ against the pump parameter A_g as well as a_l are the same as that of the normalized variance $\lambda_2(0)$, so we need not repeat the discussions here.

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