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The equivalence between B-W and K-G and R-S equations*

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The equivalence between the Bargmann–Wigner (B-W) equations and the Klein–Gordon (K-G) equations for integral spin, and the Rarita–Schwinger (R-S) equations for half integral spin is established by explicit derivation, starting from the lowest spin cases. It is demonstrated that all the constraints or subsidiary conditions imposed on the K-G or R-S equations are included in the B-W equations.

Keywords: higher spins, Bargmann–Wigner equations, Klein–Gordon equations, Rarita–Schwinger equations
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1. Introduction

In the area of high-energy physics, in order to calculate the relevant Feynman diagrams and perform amplitude analyses for high-energy processes, it is necessary to employ higher-spin relativistic wave equations and their solutions.[1−3] Among a variety of higher-spin equations available,[4−8] the Bargmann–Wigner (B-W) equations have been regarded as the simplest and least restrictive (though in many ways the most profound) set of equations.[9] In view of solution, however, the Klein–Gordon (K-G) equations for integral spins and the Rarita–Schwinger (R-S) equations for half integral spins are the most convenient equations.[10−13] The principal purpose of this paper is to show that the B-W equations are equivalent to the K-G equations for integral spins and to R-S equations for half integral spins. For lower spin cases, this equivalence is well known.[14] We are here providing an explicit derivation for a general case of an arbitrary spin, starting from the lowest spin cases. A profound property of this derivation is to demonstrate clearly that all the subsidiary conditions imposed on the K-G or R-S equations are included in the B-W equations.

2. The equivalence between B-W and K-G equations for integral spin

2.1. Spin 1

We start from the lowest spin case of spin 1, the B-W equations for this case are

$$(\partial\!\!\!/ + m)_{\alpha\alpha'}\,\Psi_{\alpha'\beta}(x) = 0, \tag{1a}$$

$$(\partial\!\!\!/ + m)_{\beta\beta'}\,\Psi_{\alpha\beta'}(x) = 0, \tag{1b}$$

where $\Psi_{\alpha\beta}(x)$ is a symmetric spinor of rank 2 and can be expressed as[14]

$$\Psi_{\alpha\beta}(x) = (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta}A^\nu(x), \tag{2}$$

with $C = \gamma_2\gamma_4$ being the charge conjugation matrix, $\gamma_\mu C$ and $\Sigma_{\mu\nu}C$ are the symmetric matrices, and $A^\nu(x)$ is the vector fields satisfying the well-known K-G equation and subsidiary condition

$$(\Box - m^2)A^\nu(x) = 0 \qquad (\nu = 1,2,3,4), \tag{3a}$$

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$$\partial_\nu A^\nu(x) = 0. \tag{3b}$$

2.2. Spin 2

The B-W equation for spin-2 reads

$$(\not\partial + m)_{\alpha\alpha'} \Psi_{\alpha'\beta\delta\tau}(x) = 0, \tag{4a}$$

$$(\not\partial + m)_{\beta\beta'} \Psi_{\alpha\beta'\delta\tau}(x) = 0, \tag{4b}$$

$$(\not\partial + m)_{\delta\delta'} \Psi_{\alpha\beta\delta'\tau}(x) = 0, \tag{4c}$$

$$(\not\partial + m)_{\tau\tau'} \Psi_{\alpha\beta\delta\tau'}(x) = 0, \tag{4d}$$

where $\Psi_{\alpha\beta\delta\tau}(x)$ is a completely symmetric multispinor of rank 4. Comparing Eqs.(4a) and (4b) with Eqs.(1a) and (1b), it is easy to see that they satisfy the same Dirac equation with respect to the indices α and β; furthermore, $\Psi_{\alpha\beta\delta\tau}(x)$ is symmetric between α and β, therefore, by using exactly the same procedure in which Eqs.(2) and (3) are derived, we obtain

$$\Psi_{\alpha\beta\delta\tau}(x) = (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu} C\partial_\mu)_{\alpha\beta} A^\nu_{\delta\tau}(x), \tag{5}$$

with $A^\nu_{\delta\tau}(x)$ satisfying the following equations

$$(\Box - m^2) A^\nu_{\delta\tau}(x) = 0 \qquad (\nu = 1,2,3,4), \tag{6a}$$

$$\partial_\nu A^\nu_{\delta\tau}(x) = 0. \tag{6b}$$

Since $\Psi_{\alpha\beta\delta\tau}(x)$ is also satisfying Dirac equation (4c), we have

$$(\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu} C\partial_\mu)_{\alpha\beta} (\not\partial + m)_{\delta\delta'} A^\nu_{\delta'\tau}(x) = 0.$$

Because α and β are arbitrary indices, this equation can be written in a matrix form, namely

$$\begin{aligned} &\mathrm{i}m(\gamma_\nu C)(\not\partial + m)_{\delta\delta'} A^\nu_{\delta'\tau}(x) \\ &+ (\Sigma_{\mu\nu} C)\partial_\mu (\not\partial + m)_{\delta\delta'} A^\nu_{\delta'\tau}(x) = 0, \end{aligned}$$

and by virtue of the independence of matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu} C$, the above equation gives

$$(\not\partial + m)_{\delta\delta'} A^\nu_{\delta'\tau}(x) = 0. \tag{7a}$$

Similarly, since $\Psi_{\alpha\beta\delta\tau}(x)$ is also satisfying Dirac equation (4d), we have

$$(\not\partial + m)_{\tau\tau'} A^\nu_{\delta\tau'}(x) = 0. \tag{7b}$$

Equations (7a) and (7b) are similar to Eqs.(1a) and (1b) with respect to the spinor indices δ and τ, therefore, we have (refer to Eqs.(2) and (3))

$$A^\nu_{\delta\tau}(x) = (\mathrm{i}m\gamma_{\nu'} C + \Sigma_{\mu'\nu'} C\partial_{\mu'})_{\delta\tau} A^{\nu\nu'}(x), \tag{8}$$

with $A^{\nu\nu'}(x)$ satisfying equations bellow

$$(\Box - m^2) A^{\nu\nu'}(x) = 0 \qquad (\nu, \nu' = 1,2,3,4), \tag{9a}$$

$$\partial_{\nu'} A^{\nu\nu'}(x) = 0. \tag{9b}$$

Substituting Eq.(8) into Eq.(6b) and applying the independence of matrices $\gamma_{\nu'} C$ and $\Sigma_{\mu'\nu'} C$, we obtain

$$\partial_\nu A^{\nu\nu'}(x) = 0. \tag{9c}$$

If Eq.(8) is substituted into Eq.(6a), one obtains the same equation as Eq.(9a). Combining Eqs.(5) and (8), $\Psi_{\alpha\beta\delta\tau}(x)$ can now be expressed as

$$\begin{aligned} \Psi_{\alpha\beta\delta\tau}(x) =& (\mathrm{i}m\gamma_\nu C + \Sigma_{\mu\nu} C\partial_\mu)_{\alpha\beta} \\ &\times (\mathrm{i}m\gamma_{\nu'} C + \Sigma_{\mu'\nu'} C\partial_{\mu'})_{\delta\tau} A^{\nu\nu'}(x). \end{aligned} \tag{10}$$

The right-hand side of this expression is symmetric both in α and β, and in δ and τ. In order to make sure that this expression is completely symmetric in all the four indices α, β, δ and τ, we require that the contraction of $\Psi_{\alpha\beta\delta\tau}(x)$ with the three independent antisymmetric Dirac matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ with respect to indices β and δ vanish, namely

$$\Psi_{\alpha\beta\delta\tau}(x)(C^{-1})_{\beta\delta} = 0, \tag{11a}$$

$$\Psi_{\alpha\beta\delta\tau}(x)(C^{-1}\gamma_5)_{\beta\delta} = 0, \tag{11b}$$

$$\Psi_{\alpha\beta\delta\tau}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta\delta} = 0. \tag{11c}$$

These, as shown bellow, will yield some new conditions. Expanding Eq.(10) and substituting it into Eqs.(11a) and (11b), using the formulae for the products of γ matrices and using Eqs.(9b) and (9c), Eqs.(11a) and (11b) become

$$(\Box - m^2) A^{\nu\nu}(x) + \mathrm{i}(\Box - m^2) A^{\nu\nu'}(x) \Sigma_{\nu\nu'} = 0, \tag{12a}$$

$$\begin{aligned} &(\Box + m^2) A^{\nu\nu}(x) + \frac{\mathrm{i}}{2}(\Box - m^2) \\ &\times (A^{\nu\nu'}(x) - A^{\nu'\nu}(x)) \Sigma_{\nu\nu'} + 2m\partial_\mu A^{\nu\nu}(x)\gamma_\mu \\ &+ 2m\varepsilon_{\mu\nu\nu'\lambda}\partial_\mu A^{\nu\nu'}(x)\gamma_5\gamma_\lambda = 0. \end{aligned} \tag{12b}$$

By the independence of γ matrices, Eq.(12a) is equivalent to Eq.(9a), but Eq.(12b) yields new conditions. The first two terms of Eq.(12b) gives

$$(\Box + m^2) A^{\nu\nu}(x) = 0, \tag{13a}$$

$$(\Box + m^2)(A^{\nu\nu'}(x) - A^{\nu'\nu}(x)) = 0. \tag{13b}$$

On the other hand, Eq.(9a) can be rewritten as

$$(\Box - m^2) A^{\nu\nu}(x) = 0, \tag{14a}$$

$$(\Box - m^2)(A^{\nu\nu'}(x) - A^{\nu'\nu}(x)) = 0. \tag{14b}$$

Combining these four equations yields the following two conditions

$$A^{\nu\nu}(x)=0, \quad A^{\nu\nu'}(x)=A^{\nu'\nu}(x). \tag{15}$$

The last equation of Eq.(15) indicates that $A^{\nu\nu'}(x)$ is symmetric in ν and ν'. From Eq.(15), it is easy to see that the last two terms of Eq.(12b) vanish. Similarly, by further using Eq.(15), Eq.(11c) gives $(\Box - m^2)A^{\nu\nu'}(x)\gamma_\nu = 0$ which is again equivalent to Eq.(9a).

Collecting all the above results, we obtain

$$\begin{aligned}\Psi_{\alpha\beta\delta\tau}(x) =&(\mathrm{i}m\gamma_{\nu_1}C+\Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta}\\ &\times(\mathrm{i}m\gamma_{\nu_2}C+\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}A^{\nu_1\nu_2}(x),\end{aligned} \tag{16}$$

with $A^{\nu_1\nu_2}(x)$ being the second rank tensor fields that satisfy the following K-G equation and subsidiary conditions

$$(\Box - m^2)A^{\nu_1\nu_2}(x)=0, \tag{17a}$$

$$\partial_{\nu_1}A^{\nu_1\nu_2}(x)=0, \quad \partial_{\nu_2}A^{\nu_1\nu_2}(x)=0, \tag{17b}$$

$$A^{\nu\nu}(x)=0, \tag{17c}$$

$$A^{\nu_1\nu_2}(x)=A^{\nu_2\nu_1}(x). \tag{17d}$$

2.3. Spin 3

The B-W equations for spin 3 read

$$(\partial\!\!\!/+m)_{\alpha_1\alpha_1'}\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)=0, \tag{18a}$$

$$(\partial\!\!\!/+m)_{\beta_1\beta_1'}\Psi_{\alpha_1\beta_1'\alpha_2\beta_2\alpha_3\beta_3}(x)=0, \tag{18b}$$

$$(\partial\!\!\!/+m)_{\alpha_2\alpha_2'}\Psi_{\alpha_1\beta_1\alpha_2'\beta_2\alpha_3\beta_3}(x)=0, \tag{18c}$$

$$(\partial\!\!\!/+m)_{\beta_2\beta_2'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2'\alpha_3\beta_3}(x)=0, \tag{18d}$$

$$(\partial\!\!\!/+m)_{\alpha_3\alpha_3'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3'\beta_3}(x)=0, \tag{18e}$$

$$(\partial\!\!\!/+m)_{\beta_3\beta_3'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3'}(x)=0, \tag{18f}$$

where $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)$ is a completely symmetric multispinor of rank 6. In exactly the same way as used in the case of spin 2, Eqs.(18a)–(18d) result in

$$\begin{aligned}&\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)\\ =&(\mathrm{i}m\gamma_{\nu_1}C+\Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha_1\beta_1}\\ &\times(\mathrm{i}m\gamma_{\nu_2}C+\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x),\end{aligned} \tag{19}$$

with $A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x)$ satisfying equations below

$$(\Box - m^2)A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x)=0, \tag{20a}$$

$$\partial_{\nu_1}A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x)=0, \quad \partial_{\nu_2}A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x)=0, \tag{20b}$$

$$A^{\nu\nu}_{\alpha_3\beta_3}(x)=0, \tag{20c}$$

$$A^{\nu_1\nu_2}_{\alpha_3\beta_3}(x)=A^{\nu_2\nu_1}_{\alpha_3\beta_3}(x). \tag{20d}$$

Substituting Eq.(19) to Eqs.(18e) and (18f), and with the aid of the independence of matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, we have

$$\begin{aligned}&(\partial\!\!\!/+m)_{\alpha_3\alpha_3'}A^{\nu_1\nu_2}_{\alpha_3'\beta_3}(x)=0,\\ &(\partial\!\!\!/+m)_{\beta_3\beta_3'}A^{\nu_1\nu_2}_{\alpha_3\beta_3'}(x)=0.\end{aligned} \tag{21}$$

Again, they are similar to Eqs.(1a) and (1b) concerning the spinor indices α_3 and β_3; furthermore, $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)$ is symmetric in α_3 and β_3, thus we have (refer to Eqs.(2) and (3))

$$A^{\nu_2\nu_2}_{\alpha_3\beta_3}(x)=(\mathrm{i}m\gamma_{\nu_3}C+\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3}A^{\nu_1\nu_2\nu_3}(x), \tag{22}$$

with $A^{\nu_1\nu_2\nu_3}(x)$ satisfying the following equations

$$(\Box - m^2)A^{\nu_1\nu_2\nu_3}(x)=0 \qquad (\nu_1,\nu_2,\nu_3=1,2,3,4), \tag{23a}$$

$$\partial_{\nu_3}A^{\nu_1\nu_2\nu_3}(x)=0. \tag{23b}$$

Substituting Eq.(22) to Eqs.(20b) through (20d), and utilizing the independence of matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, it is found

$$\partial_{\nu_1}A^{\nu_1\nu_2\nu_3}(x)=0, \quad \partial_{\nu_2}A^{\nu_1\nu_2\nu_3}(x)=0, \tag{23c}$$

$$A^{\nu\nu\nu_3}(x)=0, \quad A^{\nu_1\nu_2\nu_3}(x)=A^{\nu_2\nu_1\nu_3}(x). \tag{23d}$$

Combining Eqs.(22) and (19) yields

$$\begin{aligned}&\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)\\ =&\prod_{j=1}^{3}(\mathrm{i}m\gamma_{\nu_j}C+\Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}A^{\nu_1\nu_2\nu_3}(x).\end{aligned}$$

The right-hand side of this expression is symmetric both in $\alpha_1\beta_1\alpha_2\beta_2$, and in α_3 and β_3. In order to guarantee that this expression is completely symmetric in all the six indices $\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3$, we further require that the right-hand side of it is also symmetric in $\alpha_2\beta_2\alpha_3\beta_3$, and this is true if the contraction of $(\mathrm{i}m\gamma_{\nu_2}C+\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}(\mathrm{i}m\gamma_{\nu_3}C+\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3}A^{\nu_1\nu_2\nu_3}(x)$, with the three independent antisymmetric Driac matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ concerning indices β_2 and α_3 vanish, namely

$$\begin{aligned}&(\mathrm{i}m\gamma_{\nu_2}C+\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}\\ &\times(\mathrm{i}m\gamma_{\nu_3}C+\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3}\\ &\times A^{\nu_1\nu_2\nu_3}(x)(C^{-1})_{\beta_2\alpha_3}=0,\end{aligned}$$

$$\begin{aligned}&(\mathrm{i}m\gamma_{\nu_2}C+\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}\\ &\times(\mathrm{i}m\gamma_{\nu_3}C+\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3}\\ &\times A^{\nu_1\nu_2\nu_3}(x)(C^{-1}\gamma_5)_{\beta_2\alpha_3}=0,\end{aligned}$$

$$(\mathrm{i}m\gamma_{\nu_2}C+\Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\alpha_2\beta_2}\times(\mathrm{i}m\gamma_{\nu_3}C+\Sigma_{\mu_3\nu_3}C\partial_{\mu_3})_{\alpha_3\beta_3}\times A^{\nu_1\nu_2\nu_3}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta_2\alpha_3}=0.$$

These equations are analogous to Eqs.(11a)–(11c) and thus can be expanded in a similar way. The results indicate that only the second equation yields two new conditions

$$A^{\nu_1\nu\nu}(x)=0,\quad A^{\nu_1\nu_2\nu_3}(x)=A^{\nu_1\nu_3\nu_2}(x). \tag{23e}$$

Thus we have, for spin 3,

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x)=\prod_{j=1}^{3}(\mathrm{i}m\gamma_{\nu_j}C+\Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}A^{\nu_1\nu_2\nu_3}(x), \tag{24}$$

with $A^{\nu_1\nu_2\nu_3}(x)$ being the third rank tensor fields that satisfy the following K-G equation and subsidiary conditions

$$(\Box-m^2)A^{\nu_1\nu_2\nu_3}(x)=0\qquad(\nu_1,\nu_2,\nu_3=1,2,3,4), \tag{25a}$$

$$\partial_{\nu_1}A^{\nu_1\nu_2\nu_3}(x)=0,\quad \partial_{\nu_2}A^{\nu_1\nu_2\nu_3}(x)=0,\quad \partial_{\nu_3}A^{\nu_1\nu_2\nu_3}(x)=0, \tag{25b}$$

$$A^{\nu\nu\nu_3}(x)=0,\quad A^{\nu_1\nu\nu}(x)=0, \tag{25c}$$

$$A^{\nu_1\nu_2\nu_3}(x)=A^{\nu_2\nu_1\nu_3}(x),\quad A^{\nu_1\nu_2\nu_3}(x)=A^{\nu_1\nu_3\nu_2}(x). \tag{25d}$$

2.4. Arbitrary integral spin n

The B-W equations for spin n read

$$(\partial\!\!\!/+m)_{\alpha_1\alpha_1'}\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}(x)=0, \tag{26a}$$

$$\cdots\cdots$$

$$(\partial\!\!\!/+m)_{\beta_n\beta_n'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n'}(x)=0. \tag{26b}$$

Extending the procedure used to deal with the B-W equation for spin 1, 2 and 3, we find that the B-W wavefunctions for spin n take the form

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n}(x)=\prod_{i=1}^{n}(\mathrm{i}m\gamma_{\nu_i}C+\Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i}A^{\nu_1\nu_2\cdots\nu_n}(x), \tag{27}$$

with $A^{\nu_1\nu_2\cdots\nu_n}(x)$ satisfying the following K-G equation and subsidiary conditions

$$(\Box-m^2)A^{\nu_1\nu_2\cdots\nu_n}(x)=0\quad(\nu_1,\nu_2,\cdots,\nu_n=1,2,3,4), \tag{28a}$$

$$\partial_{\nu_i}A^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x)=0\quad(i=1,2,\cdots,n), \tag{28b}$$

$$A^{\cdots\nu\nu\cdots}(x)=0, \tag{28c}$$

$$A^{\cdots\nu_i\nu_{i+1}\cdots}(x)=A^{\cdots\nu_{i+1}\nu_i\cdots}(x)\quad(i=1,2,\cdots,n-1). \tag{28d}$$

To prove this, we make use of the induction method. This result is obviously correct for spin 1,2 and 3. Suppose it is also correct for spin n, then we consider the case of spin n+1. The B-W equations for spin n+1 are

$$(\partial\!\!\!/+m)_{\alpha_1\alpha_1'}\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}}(x)=0, \tag{29a}$$

$$\cdots\cdots$$

$$(\partial\!\!\!/+m)_{\beta_n\beta_n'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n'\alpha_{n+1}\beta_{n+1}}(x)=0, \tag{29b}$$

$$(\partial\!\!\!/+m)_{\alpha_{n+1}\alpha_{n+1}'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}'\beta_{n+1}}(x)=0, \tag{29c}$$

$$(\partial\!\!\!/+m)_{\beta_{n+1}\beta_{n+1}'}\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}'}(x)=0. \tag{29d}$$

Utilizing the results for spin n, that is Eqs.(27) and (28), we have from Eqs.(29a)–(29b)

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}}(x)=\prod_{i=1}^{n}(\mathrm{i}m\gamma_{\nu_i}C+\Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i}A^{\nu_1\nu_2\cdots\nu_n}_{\alpha_{n+1}\beta_{n+1}}(x), \tag{30}$$

with $A^{\nu_1\nu_2\cdots\nu_n}_{\alpha_{n+1}\beta_{n+1}}(x)$ satisfying equations below

$$(\Box-m^2)A^{\nu_1\nu_2\cdots\nu_n}_{\alpha_{n+1}\beta_{n+1}}(x)=0\ (\nu_1,\nu_2,\cdots,\nu_n=1,2,3,4), \tag{31a}$$

$$\partial_{\nu_i}A^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}_{\alpha_{n+1}\beta_{n+1}}(x)=0\quad(i=1,2,\cdots,n), \tag{31b}$$

$$A^{\cdots\nu\nu\cdots}_{\alpha_{n+1}\beta_{n+1}}(x)=0, \tag{31c}$$

$$A^{\cdots\nu_i\nu_{i+1}\cdots}_{\alpha_{n+1}\beta_{n+1}}(x)=A^{\cdots\nu_{i+1}\nu_i\cdots}_{\alpha_{n+1}\beta_{n+1}}(x)\quad(i=1,2,\cdots,n-1). \tag{31d}$$

Since $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}}(x)$ also satisfies Dirac equations with respect to indices α_{n+1} and β_{n+1}, substituting Eq.(30) into Eqs.(29c) and (29d) yields

$$(\partial\!\!\!/+m)_{\alpha_{n+1}\alpha_{n+1}'}A^{\nu_1\nu_2\cdots\nu_n}_{\alpha_{n+1}'\beta_{n+1}}(x)=0,\quad(\partial\!\!\!/+m)_{\beta_{n+1}\beta_{n+1}'}A^{\nu_1\nu_2\cdots\nu_n}_{\alpha_{n+1}\beta_{n+1}'}(x)=0, \tag{32}$$

which are similar to Eqs.(1a) and (1b) with respect to the spinor indices α_{n+1} and β_{n+1}. Thus we have (refer to Eqs.(2) and (3))

$$A^{\nu_1\nu_2\cdots\nu_n}_{\alpha_{n+1}\beta_{n+1}}(x)=(\mathrm{i}m\gamma_{\nu_{n+1}}C+\Sigma_{\mu_{n+1}\nu_{n+1}}C\partial_{\mu_{n+1}})_{\alpha_{n+1}\beta_{n+1}}\times A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x), \tag{33}$$

with $A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)$ satisfying the following equations

$$(\Box-m^2)A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)=0$$

$$(\nu_1,\nu_2,\cdots,\nu_n,\nu_{n+1}=1,2,3,4), \tag{34a}$$

$$\partial_{n+1}A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)=0. \tag{34b}$$

Substituting Eq.(33) into Eqs.(31b)–(31d), we obtain

$$\partial_{\nu_i}A^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n\nu_{n+1}}(x)=0 \quad (i=1,2,\cdots,n), \tag{34c}$$

$$A^{\cdots\nu\nu\cdots\nu_{n+1}}(x)=0, \tag{34d}$$

$$A^{\cdots\nu_i\nu_{i+1}\cdots}(x)=A^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i=1,2,\cdots,n-1). \tag{34e}$$

Combining Eqs.(30) and (33) yields

$$\begin{aligned}&\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}}(x)\\&=\prod_{i=1}^{n+1}(im\gamma_{\nu_i}C+\Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i}\\&\quad\times A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x).\end{aligned} \tag{35}$$

The right-hand side of this expression is symmetric both in $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$, and in $\alpha_{n+1}\beta_{n+1}$. In order to guarantee that $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}}(x)$ is completely symmetric in all the indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}$, we further require that the right-hand side of Eq.(35) is also symmetric in $\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}$, and this can be satisfied if the contraction of the factor, $(im\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}(im\gamma_{\nu_{n+1}}C+\Sigma_{\mu_{n+1}\nu_{n+1}}C\partial_{\mu_{n+1}})_{\alpha_{n+1}\beta_{n+1}}A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)$, with the three independent antisymmetric Dirac matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ concerning indices β_n and α_{n+1}, vanish, namely

$$\begin{aligned}&(im\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}\\&\times(im\gamma_{\nu_{n+1}}C+\Sigma_{\mu_{n+1}\nu_{n+1}}C\partial_{\mu_{n+1}})_{\alpha_{n+1}\beta_{n+1}}\\&\times A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)(C^{-1})_{\beta_n\alpha_{n+1}}=0,\end{aligned}$$

$$\begin{aligned}&(im\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}\\&\times(im\gamma_{\nu_{n+1}}C+\Sigma_{\mu_{n+1}\nu_{n+1}}C\partial_{\mu_{n+1}})_{\alpha_{n+1}\beta_{n+1}}\\&\times A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)(C^{-1}\gamma_5)_{\beta_n\alpha_{n+1}}=0,\end{aligned}$$

$$\begin{aligned}&(im\gamma_{\nu_n}C+\Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}\\&\times(im\gamma_{\nu_{n+1}}C+\Sigma_{\mu_{n+1}\nu_{n+1}}C\partial_{\mu_{n+1}})_{\alpha_{n+1}\beta_{n+1}}\\&\times A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta_n\alpha_{n+1}}=0.\end{aligned}$$

These equations are again analogous to Eqs.(11a)–(11c), and thus can be expanded in a similar way. The results are that only the second equation yields two new conditions

$$A^{\nu_1\nu_2\cdots\nu_{n-2}\nu\nu}(x)=0,$$

$$A^{\nu_1\nu_2\cdots\nu_{n-2}\nu_n\nu_{n+1}}(x)=A^{\nu_1\nu_2\cdots\nu_{n-2}\nu_{n+1}\nu_n}(x). \tag{36}$$

Collecting all the results above, we obtain

$$\begin{aligned}&\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\alpha_{n+1}\beta_{n+1}}(x)\\&=\prod_{i=1}^{n+1}(im\gamma_{\nu_i}C+\Sigma_{\mu_i\nu_i}C\partial_{\mu_i})_{\alpha_i\beta_i}\\&\quad\times A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x),\end{aligned} \tag{37}$$

with $A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)$ satisfying equations below

$$(\Box-m^2)A^{\nu_1\nu_2\cdots\nu_n\nu_{n+1}}(x)=0, \tag{38a}$$

$$\partial_{\nu_i}A^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n\nu_{n+1}}(x)=0 \quad (i=1,2,\cdots,n+1), \tag{38b}$$

$$A^{\cdots\nu\nu\cdots}(x)=0, \tag{38c}$$

$$A^{\cdots\nu_i\nu_{i+1}\cdots}(x)=A^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i=1,2,\cdots,n). \tag{38d}$$

Thus the conclusion expressed by Eqs.(27) and (28) for spin n has been proved also correct for spin $n+1$.

3. The equivalence between B-W and R-S equations for half-integral spin

3.1. Spin 3/2

The B-W equations for spin 3/2 are

$$(\partial\!\!\!/+m)_{\alpha\alpha'}\,\Psi_{\alpha'\beta\rho}(x)=0, \tag{39a}$$

$$(\partial\!\!\!/+m)_{\beta\beta'}\,\Psi_{\alpha\beta'\rho}(x)=0, \tag{39b}$$

$$(\partial\!\!\!/+m)_{\rho\rho'}\,\Psi_{\alpha\beta\rho'}(x)=0, \tag{39c}$$

where $\Psi_{\alpha\beta\rho}(x)$ is a completely symmetric multispinor of rank 3 and can be expressed as[13]

$$\Psi_{\alpha\beta\rho}(x)=(im\gamma_\nu C+\Sigma_{\mu\nu}C\partial_\mu)_{\alpha\beta}\,\Psi^\nu_\rho(x), \tag{40}$$

with $\Psi^\nu_\rho(x)$ being the vector-spinor satisfying the R-S equation and subsidiary conditions (the spinor index ρ is to be suppressed)

$$(\Box-m^2)\,\Psi^\nu(x)=0, \tag{41a}$$

$$\partial_\nu\,\Psi^\nu(x)=0, \tag{41b}$$

$$(\partial\!\!\!/+m)\,\Psi^\nu(x)=0, \tag{41c}$$

$$\gamma_\nu\,\Psi^\nu(x)=0. \tag{41d}$$

3.2. Spin 5/2

The B-W equations for spin 5/2 read

$$(\partial\!\!\!/+m)_{\alpha\alpha'}\,\Psi_{\alpha'\beta\delta\tau\rho}(x)=0, \tag{42a}$$

$$(\partial\!\!\!/ + m)_{\beta\beta'}\,\Psi_{\alpha\beta'\delta\tau\rho}(x) = 0, \tag{42b}$$

$$(\partial\!\!\!/ + m)_{\delta\delta'}\,\Psi_{\alpha\beta\delta'\tau\rho}(x) = 0, \tag{42c}$$

$$(\partial\!\!\!/ + m)_{\tau\tau'}\,\Psi_{\alpha\beta\delta\tau'\rho}(x) = 0, \tag{42d}$$

$$(\partial\!\!\!/ + m)_{\rho\rho'}\,\Psi_{\alpha\beta\delta\tau\rho'}(x) = 0, \tag{42e}$$

where $\Psi_{\alpha\beta\delta\tau\rho}(x)$ is a completely symmetric multi-spinor of rank 5. Comparing Eqs.(42a)–(42d) with Eqs.(4a)–(4d), it is easy to see that they satisfy the same Dirac equation with respect to the indices $\alpha\beta\delta\tau$; furthermore, $\Psi_{\alpha\beta\delta\tau\rho}(x)$ is symmetric in $\alpha\beta\delta\tau$, therefore, by using exactly the same procedure which results in Eqs.(16) and (17), we obtain

$$\begin{aligned}\Psi_{\alpha\beta\delta\tau\rho}(x) =&(\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta}\\ &\times(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\,\Psi_{\rho}^{\nu_1\nu_2}(x),\end{aligned} \tag{43}$$

with $\Psi_{\rho}^{\nu_1\nu_2}(x)$ satisfying equations below

$$(\Box - m^2)\,\Psi_{\rho}^{\nu_1\nu_2}(x) = 0, \tag{44a}$$

$$\partial_{\nu_1}\,\Psi_{\rho}^{\nu_1\nu_2}(x) = 0, \quad \partial_{\nu_2}\,\Psi_{\rho}^{\nu_1\nu_2}(x) = 0, \tag{44b}$$

$$\Psi_{\rho}^{\nu\nu}(x) = 0, \tag{44c}$$

$$\Psi_{\rho}^{\nu_1\nu_2}(x) = \Psi_{\rho}^{\nu_2\nu_1}(x). \tag{44d}$$

Since $\Psi_{\rho}^{\nu_1\nu_2}(x)$ also satisfies the Dirac equation with respect to the index ρ, substituting Eq.(43) into Eq.(42e) yields

$$\begin{aligned}&(\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta}\\ &\times(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\\ &\times(\gamma_\lambda\partial_\lambda + m)_{\rho\rho'}\,\Psi_{\rho'}^{\nu_1\nu_2}(x) = 0;\end{aligned}$$

by virtue of the independence of the γ matrices, this equation gives

$$(\partial\!\!\!/ + m)_{\rho\rho'}\,\Psi_{\rho'}^{\nu_1\nu_2}(x) = 0. \tag{44e}$$

On the other hand, the right-hand side of Eq.(43) is symmetric in indices $\alpha\beta\delta\tau$. In order to make sure that this expression is completely symmetric in all the five indices $\alpha\beta\delta\tau\rho$, we further require the right-hand side of Eq.(43) is also symmetric in indices $\delta\tau\rho$, the condition for this requirement is that the contraction of the part of $(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\,\Psi_{\rho}^{\nu_1\nu_2}(x)$ in Eq.(43) with the three independent antisymmetric Dirac matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ concerning indices $\tau\rho$ vanish, i.e.

$$(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\,\Psi_{\rho}^{\nu_1\nu_2}(x)(C^{-1})_{\tau\rho} = 0, \tag{45a}$$

$$(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\,\Psi_{\rho}^{\nu_1\nu_2}(x)(C^{-1}\gamma_5)_{\tau\rho} = 0, \tag{45b}$$

$$(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\,\Psi_{\rho}^{\nu_1\nu_2}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\tau\rho} = 0. \tag{45c}$$

Expanding Eq.(45) and using Eqs.(44b)–(44e), we have

$$\gamma_{\nu_2}(\partial\!\!\!/ + m)\,\Psi^{\nu_1\nu_2}(x) = 0, \tag{46a}$$

$$-2m\gamma_\nu\,\Psi^{\nu\nu_2}(x) + \gamma_{\nu_2}(\partial\!\!\!/ + m)\,\Psi^{\nu_1\nu_2}(x) = 0, \tag{46b}$$

$$(\partial\!\!\!/ + m)\,\Psi^{\nu_1\nu_2}(x) = 0. \tag{46c}$$

Both Eqs.(46a) and (46c) are equivalent to Eq.(44e), the last term of Eq.(46b) vanish while the first term gives

$$\gamma_\nu\,\Psi^{\nu\nu_2}(x) = 0.$$

Combining the above results, we obtain

$$\begin{aligned}\Psi_{\alpha\beta\delta\tau\rho}(x) =&(\mathrm{i}m\gamma_{\nu_1}C + \Sigma_{\mu_1\nu_1}C\partial_{\mu_1})_{\alpha\beta}\\ &\times(\mathrm{i}m\gamma_{\nu_2}C + \Sigma_{\mu_2\nu_2}C\partial_{\mu_2})_{\delta\tau}\,\Psi_{\rho}^{\nu_1\nu_2}(x),\end{aligned} \tag{47}$$

with $\Psi_{\rho}^{\nu_1\nu_2}(x)$ being a second rank tensor-spinor satisfying the following R-S equation and subsidiary conditions (the spinor index is to be suppressed)

$$(\Box - m^2)\,\Psi^{\nu_1\nu_2}(x) = 0, \tag{48a}$$

$$\partial_{\nu_1}\,\Psi^{\nu_1\nu_2}(x) = 0, \quad \partial_{\nu_2}\,\Psi^{\nu_1\nu_2}(x) = 0, \tag{48b}$$

$$\Psi^{\nu\nu}(x) = 0, \tag{48c}$$

$$\Psi^{\nu_1\nu_2}(x) = \Psi^{\nu_2\nu_1}(x), \tag{48d}$$

$$(\partial\!\!\!/ + m)\,\Psi^{\nu_1\nu_2}(x) = 0, \tag{48e}$$

$$\gamma_\nu\,\Psi^{\nu\nu_2}(x) = 0. \tag{48f}$$

3.3. Arbitrary half-integral spin $n+1/2$

The B-W equations for spin $n+1/2$ are

$$(\partial\!\!\!/ + m)_{\alpha_1\alpha_1'}\,\Psi_{\alpha_1'\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \tag{49a}$$

$$\cdots\cdots$$

$$(\partial\!\!\!/ + m)_{\beta_n\beta_n'}\,\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n'\rho}(x) = 0, \tag{49b}$$

$$(\partial\!\!\!/ + m)_{\rho\rho'}\,\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho'}(x) = 0. \tag{49c}$$

Utilizing the results for spin n, we have from Eqs.(49a)–(49b)

$$\begin{aligned}&\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)\\ &=\prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}\\ &\quad\times\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x),\end{aligned} \tag{50}$$

with $\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ satisfying equations below

$$(\Box - m^2)\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \tag{51a}$$

$$\partial_\nu\,\Psi_\rho^{\nu\nu_2\cdots\nu_i\cdots\nu_n}(x) = 0 \quad (i = 1, 2, \cdots, n), \tag{51b}$$

$$\Psi_\rho^{\cdots\nu\nu\cdots}(x) = 0, \tag{51c}$$

$$\Psi_\rho^{\cdots\nu_i\nu_{i+1}\cdots}(x) = \Psi_\rho^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i = 1, 2, \cdots, n-1). \tag{51d}$$

Since $\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ also satisfies the Dirac equation with respect to the index ρ, substituting Eq.(50) into Eq.(49c) yields

$$\prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j} \times (\not\partial + m)_{\rho\rho'}\,\Psi_{\rho'}^{\nu_1\nu_2\cdots\nu_n}(x) = 0.$$

By virtue of the independence of the γ matrices, this equation gives

$$(\not\partial + m)_{\rho\rho'}\,\Psi_{\rho'}^{\nu_1\nu_2\cdots\nu_n}(x) = 0. \tag{51e}$$

On the other hand, the right-hand side of Eq.(50) is symmetric in indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$. In order to make sure that $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is completely symmetric in all the indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho$, we further require the right-hand side of Eq.(50) is also symmetric in indices $\alpha_n\beta_n\rho$. The condition for this requirement is that the contraction of the part of $(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}\,\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ in Eq.(50) with the three independent antisymmetric matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ concerning indices β_n and ρ vanish, i.e.

$$(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times \Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1})_{\beta_n\rho} = 0, \tag{52a}$$

$$(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times \Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1}\gamma_5)_{\beta_n\rho} = 0, \tag{52b}$$

$$(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times \Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta_n\rho} = 0. \tag{52c}$$

Expanding Eq.(52) and using Eqs.(51b)–(51d), we have

$$\gamma_{\nu_n}(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \tag{53a}$$

$$-2m\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x) + \gamma_{\nu_n}(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \tag{53b}$$

$$(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0. \tag{53c}$$

Equations (53a) and (53c) are equivalent to Eq.(51e). The last term of Eq.(53b) vanish while the first term gives

$$\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x) = 0.$$

Combining the results above, we obtain

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = \prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j} \times \Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x), \tag{54}$$

with $\Psi_\rho^{\nu_1\nu_2\cdots\nu_n}(x)$ being a rank n tensor-spinor satisfying the following R-S equation and subsidiary conditions (the spinor index is to be suppressed)

$$(\Box - m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \tag{55a}$$

$$\partial_{\nu_i}\,\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x) = 0 \quad (i = 1, 2, \cdots, n), \tag{55b}$$

$$\Psi^{\cdots\nu\nu\cdots}(x) = 0, \tag{55c}$$

$$\Psi^{\cdots\nu_i\nu_{i+1}\cdots}(x) = \Psi^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i = 1, 2, \cdots, n-1), \tag{55d}$$

$$(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \tag{55e}$$

$$\gamma_\nu\,\Psi^{\nu\nu_2\nu_3\cdots\nu_n}(x) = 0. \tag{55f}$$

4. The Lagrangian density

Finally, we give the Lagrangian density for fields with integral and half-integral spins. The Lagrangian density for fields with integral spins could be expressed as

$$L(x) = -\frac{1}{4}[\partial_\mu\overline{A}^{\nu\nu_2\cdots\nu_n}(x) - \partial_\nu\overline{A}^{\mu\nu_2\cdots\nu_n}(x)] \times [\partial_\mu A^{\nu\nu_2\cdots\nu_n}(x) - \partial_\nu A^{\mu\nu_2\cdots\nu_n}(x)] - \frac{1}{2}m^2\overline{A}^{\nu\nu_2\cdots\nu_n}(x)A^{\nu\nu_2\cdots\nu_n}(x), \tag{56}$$

where

$$\overline{A}^{\nu\nu_2\cdots\nu_n}(x) = g_{\nu_1\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_n\mu_n}(A^{\mu_1\mu_2\cdots\mu_n}(x))^*,$$

$A^{\nu_1\nu_2\cdots\nu_n}(x)$ is a symmetric tensor satisfying the condition $A^{\cdots\nu\nu\cdots}(x) = 0$. From Eq.(56), the following field equations could be easily reproduced from the Euler–Lagrange equation

$$\partial_\mu\frac{\partial L}{\partial(\partial_\mu\overline{A}^{\nu\nu_2\cdots\nu_n})} = \frac{\partial L}{\partial\overline{A}^{\nu\nu_2\cdots\nu_n}},$$

$$(\Box - m^2)A^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad \partial_\nu A^{\nu\nu_2\cdots\nu_n}(x) = 0,$$

which are in agreement with Eqs.(28).

The Lagrangian density for the fields with half-integral spins takes a quite different form, which could be expressed as

$$\begin{aligned} L(x) = & -\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)(\partial\!\!\!/ + m)\,\Psi^{\mu\nu_2\cdots\nu_n}(x) \\ & + \frac{1}{3}\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\,\Psi^{\nu\nu_2\cdots\nu_n}(x) \\ & - \frac{1}{3}\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)\gamma_\mu(\partial\!\!\!/ - m)\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x), \end{aligned} \tag{57}$$

with

$$\begin{aligned} & \overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x) \\ = & g_{\nu_1\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_n\mu_n}\gamma_2(\,\Psi^{\mu_1\mu_2\cdots\mu_n}(x))^+, \end{aligned}$$

$\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x)$ being a symmetric tensor-spinor satisfying the condition $\Psi^{\cdots\nu\cdots\nu\cdots}(x) = 0$. Substituting Eq.(57) into the Euler–Lagrange equation

$$\partial_\mu\frac{\partial L}{\partial(\partial_\mu\overline{\Psi}^{\nu\nu_2\cdots\nu_n})} = \frac{\partial L}{\partial\overline{\Psi}^{\nu\nu_2\cdots\nu_n}},$$

leads to

$$\begin{aligned} & -(\partial\!\!\!/ + m)\,\Psi^{\mu\nu_2\cdots\nu_n}(x) + \frac{1}{3}\gamma_\mu A + \frac{1}{3}\partial_\mu B \\ & - \frac{1}{3}\gamma_\mu(\partial\!\!\!/ - m)B = 0, \end{aligned} \tag{58}$$

where

$$A = \partial_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x), \quad B = \gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x). \tag{59}$$

Multiplying Eq.(58) by γ_μ gives

$$2A = mB. \tag{60}$$

Operating ∂_μ on Eq.(58), with the aid of Eq.(60), yields

$$A = 0. \tag{61}$$

From Eqs.(58)–(61), the following field equations could be reproduced

$$(\Box - m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad \partial_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x) = 0,$$

$$(\partial\!\!\!/ + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad \gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x),$$

which are in agreement with Eqs. (55).

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References

[1] Chung S U 1998 *Phys. Rev.* D **57** 431
[2] Chung S U 1993 *Phys. Rev.* D **48** 1225
[3] Filippini V *et al* 1995 *Phys. Rev.* D **51** 2247
[4] Dirac P A M 1936 *Proc. R. Soc.* A **155** 447
[5] Fierz M 1939 *Helv. Phys. Acta* **12** 3
[6] Fierz M and Pauli W 1939 *Proc. R. Soc.* A **173** 211
[7] Rarita W and Schwinger J 1941 *Phys. Rev.* **60** 61
[8] Bargmann V and Wigner E P 1948 *Proc. Natl Acad. Sci. (USA)* **34** 211
[9] Salam A *et al* 1965 *Proc. R. Soc.* A **284** 146
[10] Auvil P R and Brehm J J 1966 *Phys. Rev.* **145** 1152
[11] Chung S U 1971 *Spin Formalisms* (CERN Yellow Report No CERN 71-8, Geneva, Switzerland)
[12] Huang S Z *et al* 2001 *Acta Phys. Sin.* **50** 1456 (in Chinese)
[13] Huang S Z *et al* 2002 *Chin. Phys.* **11** 561
[14] Lurie D 1968 *Particles and Fields* (New York: Wiley) p30–38