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Global exponential convergence analysis of delayed cellular neural networks*

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Some sufficient criteria have been established to ensure the global exponential stability of delayed cellular neural networks by using an approach based on delay differential inequality. Compared with the method of Lyapunov functionals as in most previous studies, our method is simpler and more effective for a stability analysis of delayed system. Some previously established results in the literature are shown to be special cases of the present result.

Keywords: global exponential stability, delayed cellular neural networks, delay differential inequality, Lyapunov functionals

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1. Introduction

Since cellular neural networks (CNNs) were proposed in 1988, CNNs have found many applications in static images.[1−4] In order to deal with moving images, delays between signal transmission must be introduced in networks. Thus, CNNs with delays (DCNNs) were presented in Ref.[5]. It is well known that stability plays an important role in these applications. For this reason, its studies have attracted much attention. It is often desired that a neural network converges in an exponential rate to ensure fast response in the network. So it is interesting to investigate the exponential stability, particularly the estimate of exponential convergence rate. Recently, some criteria have been established for the global exponential stability of DCNNs.[6−8,12,13] But these results were presented by using Lyapunov functionals, which are complicated for dealing with variable delays and obtain the bounds on the convergence rate. In this letter, based on the method of delay differential inequality, we provide some new results on global exponential stability and estimate the exponential decay rate for DCNNs.The results established here have extended those given earlier in the literature.

2.Model description

The dynamic behaviour of a continuous time DCNN can be described by the following state equations

$$x'(t) = Ax(t) + Bf(x(t-\tau(t))) + u, \qquad (1)$$

where $x(t) = [x_1(t), \ldots, x_n(t)]^{\mathrm{T}} \in R^n$, $f(x(t-\tau)) = [f_1(x_1(t-\tau_1(t))), \ldots, f_n(x_n(t-\tau_n(t)))]^{\mathrm{T}} \in R^n$; $A = \{a_{ij}\}$, $B = \{b_{ij}\}$ represent the delayed feedback matrices, while $u = [u_1, \ldots, u_n]^{\mathrm{T}}$ is a constant input vector, and time-varying delays $\tau_j(t)$ are bounded nonnegative functions:$0 < \tau_j(t) \leq \tau_j (j = 1, 2, \ldots, n)$. The activation functions $f_i (i = 1, 2, \ldots, n)$ satisfy the following condition.

(H) Each f_i satisfies

$$|f_i(\xi_1) - f_i(\xi_2)| \leq L_i |\xi_1 - \xi_2|$$

for each $\xi_1, \xi_2 \in R$, $\xi_1 \neq \xi_2$.

This type of activation functions is clearly more general than both the usual sigmoid activation functions and the piecewise linear function(PWL): $f_i(x) = \frac{1}{2}(|x+1| - |x-1|)$, which is used in Ref.[9].

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In the following discussions, we will use the notation: for any vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n$, we let $||x||$ denote a vector norm of x, and for a matrix $A \in R^{n\times n}$, $||A||$ denotes the matrix norm of A induced by the vector norm $||x||$. $\mu(A)$ is the matrix measure of A, defined as

$$\mu(A) = \lim_{\delta\to 0^+} \frac{||I + \delta A|| - 1}{\delta},$$

where I is the identity matrix with the same dimension as A.

We assume that system (1) is supplemented with initial conditions of the form

$$x_i(s) = \phi_i(s), \qquad s \in [-\tau, 0],$$

$$i = 1, 2, \ldots, n, \quad \tau = \max_{1\le j\le n}\{\tau_j\},$$

in which $\phi_i(s)$ is continuous for $s \in [-\tau, 0]$. Suppose that Eq.(1) has a unique equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$. We denote

$$||\phi - x^*|| = \sup_{-\tau\le s\le 0} ||\phi(s)||.$$

We say that an equilibrium $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is globally exponentially stable, if there exist constants $\epsilon > 0$ and $M \ge 1$ such that

$$||x(t) - x^*|| \le M||\phi - x^*||\mathrm{e}^{-\epsilon t}, \quad t \ge 0.$$

3. Global convergence analysis

Before analysis, we need the following lemma which will be used to study the global exponential convergence of Eq.(1).

Lemma [Halanay Inequality, Ref.[10] Let α and β be constants with $0 < \beta < \alpha$. Let $x(t)$ be a continuous nonnegative function on $t \ge t_0 - \tau$ satisfying the following inequality (2) for $t \ge t_0$.

$$x^{'}(t) \le -\alpha x(t) + \beta\overline{x}(t), \tag{2}$$

where $\overline{x}(t) = \sup_{t-\tau\le s\le t}\{x(s)\}$. Then

$$x(t) \le \overline{x}(t_0)\mathrm{e}^{-\lambda(t-t_0)}, \tag{3}$$

where λ is the unique positive solution of

$$\lambda = \alpha - \beta\mathrm{e}^{\lambda\tau}. \tag{4}$$

Theorem For any matrix norm $||\cdot||$, if $||B|| \le -\dfrac{\mu(A)}{||L||}$, where $L = \mathrm{diag}(L_1, L_2, \ldots, L_n)$, then the equilibrium x^* of system (1) is globally exponentially stable for any given time delay τ. In particular, the bound on exponential convergence rate $\lambda > 0$ is the unique positive solution of the following nonlinear equation

$$\lambda = -\mu(A) - ||L||||B||\mathrm{e}^{\lambda\tau}.$$

Proof Let $y(t) = x(t) - x^*$, rewrite Eq.(1) as

$$y^{'}(t) = Ay(t) + B\left[f(y(t-\tau(t)) + x^*) - f(x^*)\right]. \tag{5}$$

From Eq.(5), we have

$$\begin{aligned}
&\frac{\mathrm{d}||y(t)||}{\mathrm{d}t} - \mu(A)||y(t)|| \\
&= \lim_{\delta\to 0^+}\frac{1}{\delta}\left[||y(t+\delta)|| - ||I + \delta A||||y(t)||\right] \\
&\le \lim_{\delta\to 0^+}\frac{1}{\delta}\left[||y(t+\delta) - (I + \delta A)y(t)||\right] \\
&= \left\|\lim_{\delta\to 0^+}\frac{y(t+\delta) - y(t) - \delta Ay(t)}{\delta}\right\| \\
&= ||y^{'}(t) - Ay(t)|| \\
&= \|B\left[f(y(t-\tau(t)) + x^*) - f(x^*)\right]\| \\
&\le ||B||||L||||y(t-\tau(t))||.
\end{aligned} \tag{6}$$

From the previous inequality, we can obtain

$$\frac{\mathrm{d}||y(t)||}{\mathrm{d}t} \le \mu(A)||y(t)|| + ||L||||B||||\overline{y}(t)||.$$

According to the above Lemma, we have

$$-\mu(A) > ||L||||B||,$$

i.e.

$$||B|| < -\frac{\mu(A)}{||L||}.$$

Thus, the result of the theorem is obtained.

In order to compare our result with those presented previously, we assume that $-A = L = I$ (the identity matrix) in Eq.(1).

Corollary DCNN(1) is globally exponentially stable if one of the following conditions holds.

There exists a positive diagonal matrix $D = (d_1, d_2, \ldots, d_n)$ such that

1) $\max\limits_{1\le j\le n}\sum\limits_{i=1}^{n}\dfrac{d_j}{d_i}|b_{ij}| < 1$;

2) $\max\limits_{1\le i\le n}\sum\limits_{j=1}^{n}\dfrac{d_j}{d_i}|b_{ij}| < 1$;

3) $||D^{-1}BD||_2 < 1$;

4) $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\left(\dfrac{d_j}{d_i}|b_{ij}|\right)^2 < 1$;

5) $d_i(|b_{ii}| - 1) < \sum\limits_{j=1,j\ne i}^{n} d_j|b_{ij}|, \quad i = 1, 2, \ldots, n$;

6) $d_j(|b_{jj}|-1) < \sum_{i=1,i\neq j}^{n} d_i|b_{ij}|, \quad j=1,2,\ldots,n;$

7) the matrix $I-|B|$ is an M-matrix;

8) the spectral radius of the matrix $|B|$ less than 1, i.e. $\rho(|B|)<1$.

Proof Since for any matrix norm $||\cdot||$ and any nonsingular matrix $D, ||B||_D = ||D^{-1}BD||$ also defines a matrix norm. It is easy to see that conditions 1) 2) 3) and 4) corresponds to the column norm, row norm, spectral norm and Frobenius norm of the matrix $D^{-1}BD$, respectively. Moreover, we see that condition 2) is equivalent to that $(I-|B|)d>0$, in which $d=(d_1,d_2,\ldots,d_n)^{\mathrm{T}}>0$. According to the equivalent conditions of an M-matrix,[11] this implies that $I-|B|$ is a M-matrix and conditions 5)–8) are different equivalent conditions of M-matrix $I-|B|$. Thus the proof is completed.

Remark In Ref.[12], the same result as condition 1) is obtained. In Ref.[13], conclusion (i) of theorem 2 is the condition 2) in this work. A condition for global asymptotic stability in Ref.[14] coincides with our condition 3). The same condition as our criteria 7) but for global asymptotic stability in Ref.[15] was also achieved. In Ref.[16], a condition for global asymptotic stability was established which is consistent with our criteria 8).

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