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Exact solution and exotic coherent soliton structures of the (2+1)-dimensional generalized nonlinear Schrödinger equation*

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In this paper, a variable separation approach is used to obtain localized coherent structures of the $(2+1)$-dimensional generalized nonlinear Schrödinger equation: $\mathrm{i}\varphi_t-(\alpha-\beta)\varphi_{xx}+(\alpha+\beta)\varphi_{yy}-2\lambda\varphi\left[(\alpha+\beta)\left(\int_{-\infty}^{x}|\varphi|_y^2\mathrm{d}x+u_1(y,t)\right)-(\alpha-\beta)\left(\int_{-\infty}^{y}|\varphi|_x^2\mathrm{d}y+u_2(x,t)\right)\right]=0$. By applying a special Bäcklund transformation and introducing arbitrary functions of the seed solutions, the abundance of the localized structures of this model are derived. By selecting the arbitrary functions appropriately, some special types of localized excitations such as dromions, dromion lattice, breathers and instantons are constructed.

Keywords: variable separation approach, generalized nonlinear Schrödinger equation, coherent structure

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1. Introduction

Because of the wide applications of the soliton theory in many physics fields, such as the fluid dynamics, plasma physics, field theory, optics, etc,[1−4] the study of integrable models attracts much attention of many mathematicians and physicists. To search for exact and explicit solutions of a nonlinear physical model, especially for new exponentially localized structures like soliton solutions, is a very significant work. Since the advent of dromions unearthed by Botit *et al*,[5,6] Fokas and Santini,[7] there has been growing interest in $(2+1)$- and $(3+1)$-dimensional nonlinear evolution equations. Now several significant $(2+1)$ and $(3+1)$ models, such as the $(2+1)$-dimensional Kadomtsov–Petviashvili (KP) equation,[8] Davey–Stewartson (DS) equation,[9] generalized Korteweg–de Wries (KdV) equation,[10] asymmetric Nizhnik–Novikov–Veselov (NNV) equation,[11] sine-Gordon equation,[12] the $(3+1)$-dimensional KdV equation[13] and Jimbo–Miwa–Kadomtsev–Petviashvili equation[14] have been investigated and some special types of localized solutions for these models have also been obtained by means of different approaches, for example, the inverse scattering transformation, bilinear method, symmetry reductions, Bäcklund and Darboux transformations and so on.[15−23] From the above studies of 2+1 and $(3+1)$-dimensional models, one can see that there exist more abundant localized structures than in lower dimensions. This fact implies that there may exist similar or new localized coherent structures in other $(2+1)$-dimensional integrable models.

In this paper, we investigate the following $(2+1)$-dimensional generalized nonlinear Schrödinger (GNLS) equation[24,25]

$$\begin{aligned}&\mathrm{i}\varphi_t-(\alpha-\beta)\varphi_{xx}+(\alpha+\beta)\varphi_{yy}\\&-2\lambda\varphi\left[(\alpha+\beta)\left(\int_{-\infty}^{x}|\varphi|_y^2\mathrm{d}x+u_1(y,t)\right)\right.\\&\left.-(\alpha-\beta)\left(\int_{-\infty}^{y}|\varphi|_x^2\mathrm{d}y+u_2(x,t)\right)\right]=0,\end{aligned}\quad(1)$$

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where x and y are characteristic coordinates defined by

$$x = \xi + \eta, \quad y = \xi - \eta, \tag{2}$$

and $\lambda = \pm 1$, α and β are arbitrary real constants.

In the above equation, the scalar function $\varphi(x, y, t)$ is a complex function representing some physical quantity and $u_1(y, t)$ and $u_2(x, t)$ are arbitrary real-valued functions of the indicated arguments, and they constitute the so-called "boundary flows" of the system.

In order to seek the solutions of Eq. (1), we introduce the following transformation

$$L = -\lambda \int_{-\infty}^{y} |\varphi|_x^2 \mathrm{d}y + u_2(x, t), \quad L_y = -\lambda |\varphi|_x^2, \tag{3}$$

and

$$S = -\lambda \int_{-\infty}^{x} |\varphi|_y^2 \mathrm{d}x + u_1(y, t), \quad S_x = -\lambda |\varphi|_y^2, \tag{4}$$

in Eq. (1), where $u_1(y, t)$ and $u_2(x, t)$ are boundary terms, and convert it into a system with three variables

$$\mathrm{i}\varphi_t + a\varphi_{xx} + b\varphi_{yy} + 2(aL + bS)\varphi = 0, \tag{5}$$

$$L_y = -\lambda |\varphi|_y^2, \tag{6}$$

$$S_x = -\lambda |\varphi|_x^2. \tag{7}$$

Here $\beta - \alpha = a$ and $\beta + \alpha = b$. In Eqs. (5)–(7), L and S represent some real potentials. It is instructive to compare the form of Eqs. (5)–(7) with the NNV equation, which is a symmetric generalization of the $(2+1)$-dimensional KdV equation.

For simplicity, we only investigate the case of $\lambda = 1$ in Eqs. (5)–(7). If $\lambda = -1$, we can make a simple variable transformation $x \to -x$ (or $y \to -y$), then this situation will be identical with the former case.

This paper is organized as follows. In section 2, we apply the variable separation approach to solve the $(2+1)$-dimensional GNLS equation and obtain its exact solution. In section 3, we give some types of exotic localized coherent structures, such as dromions, dromion lattice, breathers and instantons, etc. Section 4 is a simple summary and discussion.

2. Exact solution of the $(2+1)$-dimensional GNLS equation from variable separation approach

To find soliton solutions of an equation, we can use different kinds of methods. One of the powerful methods is the variable separation approach, which was recently presented and successfully applied to some $(2+1)$-dimensional models.[26–33] Now we would use this method to investigate the $(2+1)$-dimensional GNLS equation. To solve the system (5)–(7), we first take the following Bäcklund transformation

$$\varphi = \frac{g}{f} + \varphi_0, \quad L = (\ln f)_{xx} + L_0, \quad S = (\ln f)_{yy} + S_0, \tag{8}$$

which can be obtained from the standard Painlevé truncated expansion, where f is a real function, g is a complex one and (φ_0, L_0, S_0) is an arbitrary seed solution. Substituting Eq. (8) directly into the system (5)–(7) and integrating Eqs. (6) and (7) once with respect to the arguments x and y, respectively, yields its bilinear form

$$\begin{aligned}&(aD_x^2 + bD_y^2 + \mathrm{i}D_t)gf + \varphi_0(aD_x^2 + bD_y^2)ff \\ &+ 2fg(aL_0 + bS_0) + 2f^2\varphi_0(aL_0 + bS_0) \\ &+ f^2(a\varphi_{0xx} + b\varphi_{0yy}) + \mathrm{i}f^2\varphi_{0t} = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&D_xD_yff + 2(gg^* + fg^*\varphi_0 + fg\varphi_0^* \\ &+ f^2\varphi_0\varphi_0^* + 2f^2(\partial_x^{-1}L_0)_y) = 0,\end{aligned} \tag{10}$$

$$\begin{aligned}&D_xD_yff + 2(gg^* + fg^*\varphi_0 + fg\varphi_0^* \\ &+ f^2q_0\varphi_0^* + 2f^2(\partial_y^{-1}S_0)_x) = 0,\end{aligned} \tag{11}$$

where $\partial_x^{-1}, \partial_y^{-1}$ are integration symbols with respect to the argument x or y, and D_x, D_y, D_t are defined as

$$\begin{aligned}&D_x^m D_y^n D_t^k ff \\ &= \lim_{x'=x, y'=y, t'=t} \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \\ &\times \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k ff,\end{aligned} \tag{12}$$

which are the usual bilinear operators introduced first by Hirota.[15]

To discuss further, we take the seed solution (φ_0, L_0, S_0) as

$$\varphi_0 = 0, \quad L_0 = u_0(x, t), \quad S_0 = v_0(y, t), \tag{13}$$

then Eqs. (5)–(7) can be simplified to

$$(aD_x^2 + bD_y^2 + \mathrm{i}D_t)gf + 2fg(au_0 + bv_0) = 0, \tag{14}$$

$$D_xD_yff + 2gg^* = 0. \tag{15}$$

In order to find some interesting solutions of Eqs. (14) and (15), we can use the variable separation ansatz

$$f = a_1 u + a_2 v + a_3 uv, \quad g = u_1 v_1 \exp(\mathrm{i}r + \mathrm{i}s), \quad (16)$$

where a_1, a_2 and a_3 are arbitrary constants; and $u \equiv u(x,t)$, $v \equiv v(y,t)$, $u_1 \equiv u_1(x,t)$, $v_1 \equiv v_1(y,t)$, $r \equiv r(x,t)$, $s \equiv s(y,t)$ are all real functions of the indicated variables. Substituting Eq. (16) into Eq. (15), we have

$$u_1^2 v_1^2 - a_1 a_2 u_x v_y = 0. \quad (17)$$

Since the functions u, u_1 are only functions of $\{x,t\}$ and the functions v, v_1 are only functions of $\{y,t\}$, Eq. (17) can be solved by the following variable separated equations

$$u_1 = \delta_1 \sqrt{a_1 a_2 c_0^{-1} u_x}, \quad (18)$$

$$v_1 = \delta_2 \sqrt{c_0 v_y} \quad (\delta_1^2 = \delta_2^2 = 1). \quad (19)$$

Similar to the above procedure, inserting Eq. (16) with Eqs. (18) and (19) into Eq. (14) and separating the real and imaginary parts of the resulting equation yields the following variable separated equations

$$u_t = -2a r_x u_x + c_1(a_2 + a_3 u)^2 + c_2(a_2 + a_3 u) + a_1 a_2 c_3, \quad (20)$$

$$v_t = -2b s_y v_y - c_3(a_1 + a_3 v)^2 - c_2(a_1 + a_3 v) - a_1 a_2 c_1, \quad (21)$$

$$4(s_t + b s_y^2 - 2b v_0) v_y^2 - 2b v_y v_{yyy} + b v_{yy}^2 + c_4 v_y^2 = 0, \quad (22)$$

$$4(r_t + a r_x^2 - 2a u_0) u_x^2 - 2a u_x u_{xxx} + a u_{xx}^2 - c_4 u_x^2 = 0. \quad (23)$$

In Eqs. (18)–(23), c_0, c_1, c_2, c_3, c_4 are all arbitrary functions of t. From Eqs. (18) and (19), we know that the real conditions of u and v require

$$a_1 a_2 c_0^{-1} u_x \geq 0, \quad (24)$$

$$c_0 v_y \geq 0. \quad (25)$$

Although it is not an easy task to obtain general solutions of Eqs. (20)–(23) for any fixed u_0 and v_0, we can treat the problem in an alternative way. Since u_0 and v_0 are arbitrary seed solutions, we can regard u and v as arbitrary functions of $\{x,t\}$ and $\{y,t\}$ under the conditions (24) and (25) respectively. The functions r and s can be expressed in terms of u and v simply by integrations from Eqs. (20) and (21). Then the seed solutions u_0 and v_0 can be fixed by Eqs. (22) and (23).

Finally, substituting Eq. (16) with Eqs. (18)–(23) into Eq. (8), we obtain a quite general solution of the $(2+1)$-dimensional system (5)–(7)

$$\varphi = \frac{\delta_1 \delta_2 \sqrt{a_1 a_2 u_x v_y} \exp(\mathrm{i}r + \mathrm{i}s)}{a_1 u + a_2 v + a_3 uv}, \quad (26)$$

$$L = u_0 + \left(\frac{a_1 u_{xx} + a_3 v u_{xx}}{a_1 u + a_2 v + a_3 uv} - \frac{(a_1 u_x + a_3 v u_x)^2}{(a_1 u + a_2 v + a_3 uv)^2} \right), \quad (27)$$

$$S = v_0 + \left(\frac{a_2 v_{yy} + a_3 u v_{yy}}{a_1 u + a_2 v + a_3 uv} - \frac{(a_2 v_y + a_3 u v_y)^2}{(a_1 u + a_2 v + a_3 uv)^2} \right), \quad (28)$$

with two arbitrary u and v under the conditions (24) and (25); and u_0, v_0 are determined by Eqs. (22) and (23). Especially, the module square of the field φ reads

$$\Phi = |\varphi|^2 = \frac{a_1 a_2 u_x v_y}{(a_1 u + a_2 v + a_3 uv)^2} \quad (29)$$

or

$$\Phi = |\varphi|^2 = \frac{a_1 a_2 U_x V_y}{4\left(A_1 \cosh \frac{1}{2}(U + V + C_1) + A_2 \cosh \frac{1}{2}(U - V + C_2) \right)^2}, \quad (30)$$

where

$$u = b_1 + \mathrm{e}^U, \quad v = b_2 + \mathrm{e}^V, \quad (31)$$

and

$$\begin{aligned} A_1 &= \sqrt{a_3(a_1 b_1 + a_2 b_2 + a_3 b_1 b_2)}, \\ A_2 &= \sqrt{(a_1 + a_3 b_2)(a_2 + a_3 b_1)}, \end{aligned} \quad (32)$$

$$\begin{aligned} C_1 &= \ln \frac{a_3}{a_1 b_1 + a_2 b_2 + a_3 b_1 b_2}, \\ C_2 &= \ln \frac{a_1 + a_3 b_2}{a_2 + a_3 b_1}, \end{aligned} \quad (33)$$

with b_1 and b_2 being arbitrary constants. U and V are also arbitrary functions of $\{x,t\}$ and $\{y,t\}$, respectively, under the conditions

$$a_1 a_2 U_x V_y \geq 0. \quad (34)$$

As pointed out in Ref.[31], as $u(x,t)$ and $v(y,t)$ are arbitrary functions of the indicated arguments, Eq. (29) (or Eq. (30)) reveals quite abundant soliton structures. From Eqs. (29) and (30), it is easy to know that for arbitrary u and v with the boundary

conditions

$$u|_{x\to-\infty}\to B_1,\ u|_{x\to+\infty}\to B_2,$$

$$v|_{y\to-\infty}\to B_3,\ v|_{y\to+\infty}\to B_4, \tag{35}$$

where B_1, B_2, B_3 and B_4 are arbitrary constants which may be infinities, there is a coherent soliton solution localized in all directions. Until now, many exotic coherent structures of Eq. (29) (or Eq. (30)) have been widely investigated within other physical models, such as Refs.[34]–[37], etc. In the following discussions we only cite some interesting examples and omit the related plots.

3. Some types of localized coherent soliton structures of the GNLS equation

3.1. Multi-dromion solutions driven by straight-line ghost solitons

Usually, as pointed out in Ref.[38], the multi-dromion solutions which are localized in all directions are driven by multiple straight-line ghost solitons with some suitable dispersion relations, and the dromions are located at the cross points of the straight lines. If we take functions u and v as

$$u=\sum_{i=1}^{M}\exp(K_ix+\Omega_it+x_{0i})+1\equiv\sum_{i=1}^{M}\exp(\xi_i), \tag{36}$$

$$v=\sum_{i=1}^{N}\exp(k_iy+\omega_it+y_{0i})\equiv\sum_{i=1}^{N}\exp(\eta_i), \tag{37}$$

where K_i, k_i, Ω_i, ω_i, x_{0i} and y_{0i} are arbitrary constants, and M and N are positive integers, then we can obtain the first type of special multi-dromion solutions. There is no dispersion relation among K_i, k_i, Ω_i and ω_i.

3.2. Multi-dromion solutions driven by straight and curved line ghost solitons

Meanwhile, we have known that for many $(2+1)$-dimensional nonlinear physical models, the dromion solutions can be driven not only by straight-line ghost solitons, but also by curved-line and straight-line solitons. From Eq. (30), we know that the dromion solutions can also be driven by curved-line and straight-line solitons here. In fact, general multi-dromion solutions of this GNLS equation expressed by Eq. (30) are driven by two sets of straight-line solitons and curved-line solitons. The first set of straight-line solitons appears in the factor V_y, say, if we take

$$V_y=\sum_{i=1}^{N}V_i(y-y_i(t)), \tag{38}$$

where $V_i=V_i(y-y_i(t))$ denotes a straight-line soliton which is finite at the line $y=y_i(t)$, and decays rapidly apart from the line. Similarly, the second set of straight-line solitons appears in the factor U_x. Finally, the curved-line solitons are determined by the factors $A_1\cosh\frac{1}{2}(U+V+C_1)$ and $A_2\cosh\frac{1}{2}(U-V+C_2)$ in Eq. (30) and the curves are determined by

$$U+V+C_1=\min(U+V+C_1),$$

$$U-V+C_2=\min(U-V+C_2), \tag{39}$$

while the number of the curved-line solitons is determined by the branches of Eq. (39).

For example, if taking

$$U=x^5+2x^3+0.2,\quad V=2y^3+0.2y+0.4, \tag{40}$$

$$a_1=a_2=a_3=b_1=b_2=1, \tag{41}$$

the corresponding curves are determined by

$$x^5+2x^3+2y^3+0.2y+0.6-\ln 3=0,$$

$$x^5+2x^3-2y^3-0.2y-0.2=0, \tag{42}$$

then we can construct a multi-dromion (four-dromion) structure for the quantity $\Phi=|\varphi|^2$ driven by the two ghost curved-line solitons. Furthermore, if taking

$$U=\sum_{n=-N}^{N}0.1\tanh(x+4n),$$

$$V=\sum_{m=-M}^{M}0.1\tanh(y+4m), \tag{43}$$

$$M=N=2,\quad a_1=a_2=a_3=b_1=b_2=1, \tag{44}$$

then we can obtain a 5×5 "dromion lattice" solution for the field $\Phi=|\varphi|^2$.

3.3. Standing and moving breather-like structures

Obviously, if some types of periodic functions of time t are included in the above types of localized solutions, then all these types of solutions will become the corresponding breathers. For example, if we take

$$U=2(1.2+\cos(2\pi t))(x^3+x+1),$$

$$V = 2(1.2 + \cos(2\pi t))(0.5y^3 + y + 0.1), \quad (45)$$

and

$$a_1 = a_2 = b_1 = b_2 = 1, \quad (46)$$

in Eq. (30), then we can obtain a special type standing breather solution. If x and y in Eq. (45) are changed to $x - \mu_1 t$ and $y - \mu_2 t$, then Eq. (30) with Eq. (45) becomes moving breather solution, which can breathe in many ways, say, in its amplitudes of peaks, shapes, distances among the peaks and even its number of peaks at different times.

3.4. Instanton-like excitations

If a further decaying factor of time t is contained in U and V of Eq. (30), say, multiplying U and V by $\mathrm{sech}(at)$, then the localized structures listed above become instanton-like solutions. For instance, if we take

$$U = 4\mathrm{sech}(4t)\tanh(x^3), \quad V = 4\tanh(1.2y), \quad (47)$$

and Eq. (46), then we can derive a standard dromion-like instanton solution (a moving instanton in direction), which decays tremendously in a short time.

3.5. On the nonexistence of the ring solitons and lumps

In some high dimensions, in addition to the point-like localized coherent excitations, there may be some other types of physically significant localized excitation.[28,29,34] Though the right-hand side of Eq. (29) (or Eq. (30)) is valid for some other models like the NNV model,[28] the $(2+1)$-dimensional KdV equation,[29], etc., just the same as the case Lou has pointed out in Ref.[31] that in the DS equation there is no ring solitons and lumps, so we cannot obtain the similar ring type of soliton solutions from Eqs. (29) and/or (30) due to the real conditions (24) and (25). It is also known that in high dimensions, like the KP equation, a special type of localized structures, called lump solutions, may also be formed by rational functions. For the GNLS system (1), we cannot obtain an analytical lump solution from Eq. (29) by selecting u and v as power or rational functions, though some types of singular rational solutions, called positons, can be found.

4. Summary and discussion

In summary, using the variable separation approach for the $(2+1)$-dimensional GNLS equation, we have obtained many types of multi-soliton solutions because of the existence of the arbitrary functions u and v. By appropriately selecting the arbitrary functions, the multiple localized solutions of Eq. (29) may be dromions, dromion lattice, breathers and instantons, etc. The breathers may breathe in their amplitudes, shapes, distances among the peaks and even the number of peaks.

It is worth to mention that all the localized coherent structures we have cited here are discussed under the real conditions (24) and (25), a wealth of other type localized coherent structures, such as ring soliton and multi-lumps solutions cannot obtained here, though it is valid from Eq. (29) (or (30)) itself to construct them. The variable separation method has been applied successfully to find localized structure solutions here. More about applying this method to other $(2+1)$-dimensional nonlinear physical models and more about the properties of the multiple localized coherent solitons are worthy of further study.

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