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An open plus nonlinear closed loop control of chaotic oscillators*

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An open plus nonlinear closed loop control law is presented for chaotic oscillations described by a set of non-autonomous second-order ordinary differential equations. It is proven that the basins of entrainment are global when the right-hand sides of the equations are given by arbitrary polynomial functions. The forced Duffing oscillator and the forced van der Pol oscillator are treated as numerical examples to demonstrate the applications of the method.

Keywords: chaos control, entrainment basin, oscillator
PACC: 0545, 4610

1. Introduction

Many dynamical systems frequently exhibit chaotic behaviour that is characterized by the sensitivity to initial conditions, unperiodicity, and intrinsic stochasticity. Such behaviour yields unpredictability in the long-time which generally reduces the usefulness of the systems. Much effort has been devoted to trying to devise methods for controlling chaos[1−11] since the pioneering work of Ott *et al.*[12] Among the many approaches,[13] there are entrainment control and its development, the open plus closed loop (OPCL) control.

Jackson[14] advanced the entrainment control, which is the development of the resonant control presented by Hubler and Luscher,[15] one of the earliest methods of controlling chaos. Its essence is to design an external excitation based on the control goal, as the open-loop control, so that the control goal is a special solution of the controlled system. To ensure the stability of this particular solution, it is required that the control goal be located in suitable regions in the phase space (called convergent regions) and that there be suitable initial conditions (called the basin of entrainment) when the control is activated.[16,17] The entrainment control has two restrictions. The system must be dissipative, because the convergent region exists only in this case. The unstable periodic orbits of the uncontrolled system cannot be the goal of the control, that is, the methods cannot be applied to solve the problems of stabilization. In addition, the convergent regions and the basin of entrainment are difficult to determine in practical applications.

Jackson and Grosu[18] advanced an OPCL control based on the entrainment control. Introducing a closed-loop term into the entrainment control law remedies its defect, and the OPCL control may be implemented in experiments.[19,20] In fact, the basins of entrainment in the OPCL control for some classic chaotic systems are global.[18,21] However, the OPCL control method still has specific limitations for more complicated and higher-order nonlinear systems, and a new control method of open plus nonlinear closed loop (OPNCL) has been proposed.[22]

The above-mentioned approaches can be employed to control nonlinear oscillators governed by non-autonomous second-order ordinary differential equations. Nevertheless, the number of control parameters required by the OPCL control or the OPNCL control for general chaotic continuous dynamical systems[18,21,22] is equal to the dimension of phase spaces. For example, two control parameters are needed to control the nonlinear oscillator with one degree of freedom

$$\ddot{x} = f(x, \dot{x}, t). \tag{1}$$

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The controlled system should take the form[18,22]

$$\begin{aligned}\dot{x}_1 =&x_2+\nu(x_1,x_2,t),\\ \dot{x}_2 =&f(x_1,x_2,t)+u(x_1,x_2,t),\end{aligned}\tag{2}$$

where x_1, x_2 are the generalized coordinate and the generalized velocity, and u, ν are controllable parameters. In some practical cases, there may not be such controllable parameters available. Chen and Liu[23] have developed an OPCL control approach that requires only one control parameter. The controlled system should take the form

$$\dot{x}_1=x_2,\quad \dot{x}_2=f(x_1,x_2,t)+u(x_1,x_2,t).\tag{3}$$

Here we investigate the control for chaos in oscillating systems with n degrees of freedom

$$\ddot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\dot{\boldsymbol{x}},t)\quad \boldsymbol{x}\in R^n.\tag{4}$$

An OPNCL control law is proposed. The entrainment capabilities of the OPNCL control are analysed. Some numerical examples of controlling chaotic oscillators are presented.

2. The open plus nonlinear closed loop control law

We consider a controllable inertial uncoupling nonlinear oscillation system with n degrees of freedom

$$\ddot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\dot{\boldsymbol{x}},t)+\boldsymbol{u},\tag{5}$$

where n dimension vectors $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are the generalized coordinate, velocity and acceleration, respectively, the n-dimensional vector $\boldsymbol{u}$ is a control parameter, and t is the time variable. Suppose the right-hand side of Eq.(5) is a polynomial of degree m, that is

$$\begin{aligned}&f_i(\boldsymbol{x},\dot{\boldsymbol{x}},t)\\ =&\sum_{\substack{j_1+\cdots+j_n\\ +k_1+\cdots+k_n=0}}^{m} a^{(i)}_{j_1\cdots j_n k_1\cdots k_n}x_1^{j_1}\cdots x_1^{j_n}\dot{x}_1^{k_1}\cdots\dot{x}_1^{k_n},\end{aligned}\tag{6}$$

where there at least exists $a^{(1)}_{j_1\cdots j_n k_1\cdots k_n}\neq 0(1\le i\le n, j_1+\cdots+j_n+k_1+\cdots+k_n=m)$. We assume that system (5) has chaotic motion if no control is applied ($\boldsymbol{u}$=$\boldsymbol{0}$).

We give the desired control goal $\boldsymbol{g}(t)$ that can be equilibrium points, periodic orbits, chaotic motions and so on. We define the control term

$$\begin{aligned}\boldsymbol{u}=&\ddot{\boldsymbol{g}}(t)-\boldsymbol{f}(\boldsymbol{g},\dot{\boldsymbol{g}},t)+\boldsymbol{C}(\boldsymbol{g},\dot{\boldsymbol{g}},t)(\boldsymbol{g}(t)-\boldsymbol{x}(t))\\ &+\boldsymbol{D}(\boldsymbol{g},\dot{\boldsymbol{g}},t)(\dot{\boldsymbol{g}}(t)-\dot{\boldsymbol{x}}(t)),\end{aligned}\tag{7}$$

where elements of $n\times n$ real matrices $\boldsymbol{C}$ and $\boldsymbol{D}$ are

$$\begin{aligned}C_{ij}=&\alpha_{ij}+\frac{\partial f_i}{\partial x_j}+\frac{1}{2!}\sum_{j_1=1}^{n}\left[\frac{\partial^2 f_i}{\partial x_j\partial x_{j_1}}(x_{j_1}-g_{j_1})\right.\\ &\left.+\frac{\partial^2 f_i}{\partial x_j\partial\dot{x}_{j_1}}(\dot{x}_{j_1}-\dot{g}_{j_1})\right]+\cdots+\frac{1}{m!}\\ &\times\sum_{j_1,\cdots,j_{m-1}=1}^{n}\left[\frac{\partial^m f_i}{\partial x_j\partial x_{j_1}\cdots\partial x_{j_{m-1}}}\right.\\ &\times(x_{j_1}-g_{j_1})\cdots(x_{j_{m-1}}-g_{j_{m-1}})\\ &+(m-1)\frac{\partial^m f_i}{\partial x_j\partial\dot{x}_{j_1}\partial x_{j_2}\cdots\partial x_{j_{m-1}}}\\ &\times(\dot{x}_{j_1}-\dot{g}_{j_1})(x_{j_2}-g_{j_2})\cdots(x_{j_{m-1}}-g_{j_{m-1}})\\ &+\frac{\partial^m f_i}{\partial x_j\partial\dot{x}_{j_1}\cdots\partial\dot{x}_{j_{m-1}}}\\ &\left.\times(\dot{x}_{j_1}-\dot{g}_{j_1})\cdots(\dot{x}_{j_{m-1}}-\dot{g}_{j_{m-1}})\right],\end{aligned}\tag{8}$$

$$\begin{aligned}D_{ij}=&\beta_{ij}+\frac{\partial f_i}{\partial\dot{x}_j}+\frac{1}{2!}\sum_{j_1=1}^{n}\left[\frac{\partial^2 f_i}{\partial\dot{x}_j\partial x_{j_1}}(x_{j_1}-g_{j_1})\right.\\ &\left.+\frac{\partial^2 f_i}{\partial\dot{x}_j\partial\dot{x}_{j_1}}(\dot{x}_{j_1}-\dot{g}_{j_1})\right]+\cdots+\frac{1}{m!}\\ &\times\sum_{j_1,\cdots,j_{m-1}=1}^{n}\left[\frac{\partial^m f_i}{\partial\dot{x}_j\partial x_{j_1}\cdots\partial x_{j_{m-1}}}\right.\\ &\times(x_{j_1}-g_{j_1})\cdots(x_{j_{m-1}}-g_{j_{m-1}})\\ &+(m-1)\frac{\partial^m f_i}{\partial\dot{x}_j\partial\dot{x}_{j_1}\partial x_{j_2}\cdots\partial x_{j_{m-1}}}\\ &\times(\dot{x}_{j_1}-\dot{g}_{j_1})(x_{j_2}-g_{j_2})\cdots(x_{j_{m-1}}-g_{j_{m-1}})\\ &+\frac{\partial^m f_i}{\partial\dot{x}_j\partial\dot{x}_{j_1}\cdots\partial\dot{x}_{j_{m-1}}}\\ &\left.\times(\dot{x}_{j_1}-\dot{g}_{j_1})\cdots(\dot{x}_{j_{m-1}}-\dot{g}_{j_{m-1}})\right].\end{aligned}\tag{9}$$

In Eqs.(8) and (9) α_{ij} and β_{ij} (i,j=1,2,$\cdots,n$) are undetermined coefficients, and all partial derivatives are evaluated at $(\boldsymbol{g},\dot{\boldsymbol{g}},t)$.

The following theorem can be proved.

Theorem: For any smooth function $\boldsymbol{g}(t)$, there exist matrices $\boldsymbol{A}=[\alpha_{ij}]$ and $B=[\beta_{ij}]$ such that the control law (7) starting at $t=t_0$ can entrain the controlled system (5) to the goal $\boldsymbol{g}(t)$, and the basins of entrainment associated with $\boldsymbol{g}$ and t_0

$$BE(\boldsymbol{g},t_0)=\{\boldsymbol{x}(t_0)|\lim_{t\to\infty}|\boldsymbol{x}(t)-\boldsymbol{g}(t)|=\boldsymbol{0}\}\tag{10}$$

are global.

In fact, to investigate the entrainment capability, one sets

$$\boldsymbol{x}(t)=\boldsymbol{g}(t)+\boldsymbol{\varepsilon}(t).\tag{11}$$

Then from Eq.(5),

$$\ddot{\boldsymbol{\varepsilon}}(t) = \boldsymbol{f}(\boldsymbol{g}+\boldsymbol{\varepsilon}, \dot{\boldsymbol{g}}+\dot{\boldsymbol{\varepsilon}}, t) + \boldsymbol{u} - \ddot{\boldsymbol{g}}(t). \tag{12}$$

Notice that $\boldsymbol{f}$ is a polynomial of degree m, hence

$$\begin{aligned}
&f_i(\boldsymbol{g}+\boldsymbol{\varepsilon}, \dot{\boldsymbol{g}}+\dot{\boldsymbol{\varepsilon}}, t) \\
=&\boldsymbol{f}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t) + \sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\varepsilon_j + \frac{\partial f_i}{\partial \dot{x}_j}\dot{\varepsilon}_j\right) \\
&+ \frac{1}{2!}\sum_{j,j_1=1}^{n}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_{j_1}}\varepsilon_j\varepsilon_{j_1} + 2\frac{\partial^2 f_i}{\partial \dot{x}_j \partial x_{j_1}}\dot{\varepsilon}_j\varepsilon_{j_1}\right. \\
&\left.+ \frac{\partial^2 f_i}{\partial \dot{x}_j \partial \dot{x}_{j_1}}\dot{\varepsilon}_j\dot{\varepsilon}_{j_1}\right) + \cdots + \frac{1}{m!} \\
&\times \sum_{j,j_1,\cdots,j_{m-1}=1}^{n}\left(\frac{\partial^m f_i}{\partial x_j \partial x_{j_1}\cdots\partial x_{j_{m-1}}}\varepsilon_j\varepsilon_{j_1}\cdots\varepsilon_{j_{m-1}}\right. \\
&+ m\frac{\partial^m f_i}{\partial \dot{x}_j \partial x_{j_1}\cdots\partial x_{j_{m-1}}}\dot{\varepsilon}_j\varepsilon_{j_1}\cdots\varepsilon_{j_{m-1}} \\
&+ \frac{m(m+1)}{2}\frac{\partial^m f_i}{\partial \dot{x}_j \partial \dot{x}_{j_1}\partial x_{j_2}\cdots\partial x_{j_{m-1}}}\dot{\varepsilon}_j\dot{\varepsilon}_{j_1}\varepsilon_{j_2}\cdots\varepsilon_{j_{m-1}} \\
&\left.+ \cdots + \frac{\partial^m f_i}{\partial \dot{x}_j \partial \dot{x}_{j_1}\cdots\partial \dot{x}_{j_{m-1}}}\dot{\varepsilon}_j\dot{\varepsilon}_{j_1}\cdots\dot{\varepsilon}_{j_{m-1}}\right).
\end{aligned} \tag{13}$$

Substituting Eqs.(7)–(9) and (13) into Eq.(12), one obtains

$$\ddot{\boldsymbol{\varepsilon}} = -\boldsymbol{A}\boldsymbol{\varepsilon} - \boldsymbol{B}\dot{\boldsymbol{\varepsilon}}. \tag{14}$$

For convenience, $\boldsymbol{A}$ and $\boldsymbol{B}$ can be taken as diagonal matrices

$$\boldsymbol{A} = \mathrm{diag}[\alpha_1, \alpha_2, \cdots, \alpha_n], \quad \boldsymbol{B} = \mathrm{diag}[\beta_1, \beta_2, \cdots, \beta_n]. \tag{15}$$

Then Eq.(14) is uncoupled into

$$\ddot{\varepsilon}_j + \beta_j\dot{\varepsilon}_j + \alpha_j\varepsilon_j = 0 \quad (j = 1, 2, \cdots, n). \tag{16}$$

The poles of system (16) may be arbitrarily placed for suitable α_j and β_j.[24] Thus, $\boldsymbol{\varepsilon}(t)$ solved from Eq.(16) asymptotically approaches zero for all $\boldsymbol{\varepsilon}(t_0)$.

The control signal determined by Eq.(7) is composed of two parts: the open loop part without feedback

$$\boldsymbol{u} = \ddot{\boldsymbol{g}}(t) - \boldsymbol{f}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t) \tag{17}$$

and the nonlinear closed loop part with feedback

$$\begin{aligned}
\boldsymbol{u} =& \boldsymbol{C}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t)(\boldsymbol{g}(t) - \boldsymbol{x}(t)) \\
&+ \boldsymbol{D}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t)(\dot{\boldsymbol{g}}(t) - \dot{\boldsymbol{x}}(t)).
\end{aligned} \tag{18}$$

Therefore Eq.(7) gives an OPNCL control law.

If nonlinear terms are omitted in Eq.(7), one can obtain the OPCL control law[23,25,26]

$$\begin{aligned}
\boldsymbol{u} =& \ddot{\boldsymbol{g}}(t) - \boldsymbol{f}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t) \\
&+ \left(\left[\frac{\partial f_i}{\partial x_j}\right] - \boldsymbol{A}\right)(\boldsymbol{g}(t) - \boldsymbol{x}(t)) \\
&+ \left(\left[\frac{\partial f_i}{\partial x_j}\right] - \boldsymbol{B}\right)(\dot{\boldsymbol{g}}(t) - \dot{\boldsymbol{x}}(t)).
\end{aligned} \tag{19}$$

The entrainment capabilities of the OPCL control of oscillators are harder to analyse than those of the OPCL control of general dynamical systems. Hence the determination of the basins of entrainment in the OPCL control approach is a very complicated problem.

It is desired that the control action (7) does not change the system abruptly. Thus, one defines a switch function rising from 0 to 1

$$S(t) = \begin{cases} 1 - \mathrm{e}^{t_0 - t}, & t \geq t_0, \\ 0, & t < t_0. \end{cases} \tag{20}$$

The switch function (20) relaxes the sharpness of response of the system[17,27,28] when the control law is activated. The OPNCL control law takes the form

$$\begin{aligned}
\boldsymbol{u} =& S(t)[\ddot{\boldsymbol{g}}(t) - \boldsymbol{f}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t) + \boldsymbol{C}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t)(\boldsymbol{g}(t) - \boldsymbol{x}(t)) \\
&+ \boldsymbol{D}(\boldsymbol{g}, \dot{\boldsymbol{g}}, t)(\dot{\boldsymbol{g}}(t) - \dot{\boldsymbol{x}}(t))].
\end{aligned} \tag{21}$$

3. Numerical examples

Now the forced Duffing oscillator with a controllable parameter[29]

$$\ddot{x} = -0.2\dot{x} + x - 0.01x^2 - x^3 + 0.3\cos t + u \tag{22}$$

is treated as the first example to demonstrate the application of the OPNCL control.

Chaotic behaviour appears in oscillator (22) without control. The Poincaré map of the uncontrolled case is shown in Fig.1. The Lyapunov exponents are 0.163, 0.000 and –0.360, and the largest Lyapunov exponent varying with time is depicted in Fig.2.

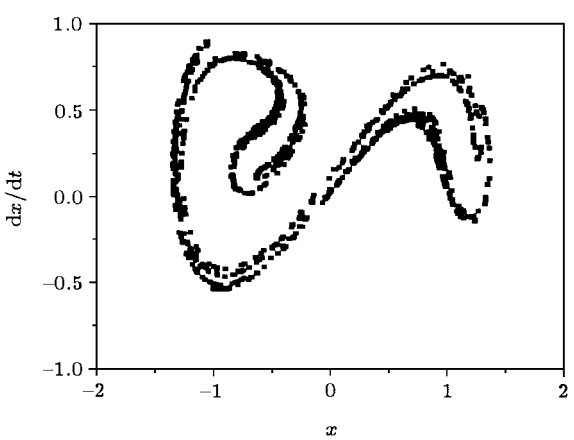


Fig.1. The Poincaré map of oscillator (22) without control.

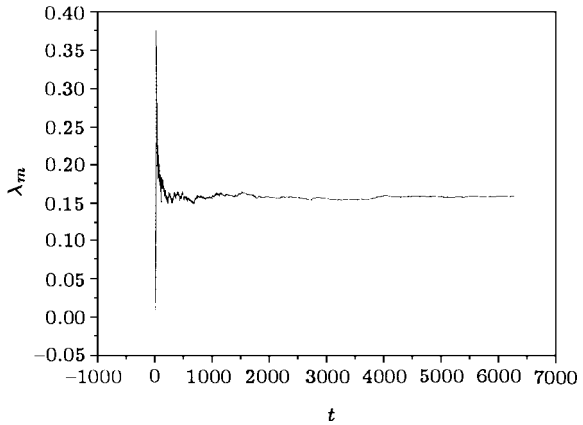


Fig.2. The largest Lyapunov exponent of oscillator (22) without control.

Letting

$$f_{\rm D}(x,\dot{x},t) = -0.2\dot{x} + x - 0.01x^2 - x^3 + 0.3\cos t, \quad (23)$$

then Eqs.(21), (8) and (9) give the OPNCL control law of system (22)

$$\begin{aligned} u =& S(t)[\ddot{g} - f_{\rm D}(g,\dot{g},t) \\ &+ (\alpha + 1 - 0.01g - 0.01x - 3gx + g^3 - 3g^2x \\ &+ 3gx^2 - x^3)(g - x) + (\beta - 0.2)(\dot{g} - \dot{x})], \quad (24) \end{aligned}$$

where $g(t)$ is the control goal. The control goals successively are a fixed point and a period-two motion

$$g(t) = 1, \quad g(t) = 0.2\sin 0.5t. \quad (25)$$

We start control at t_1=40.0, and we choose α=4.0 and β=2.8. The results are, respectively, shown in Figs.3 and 4, in which the solid and dotted lines correspond to the time histories of the controlled and uncontrolled systems, respectively.

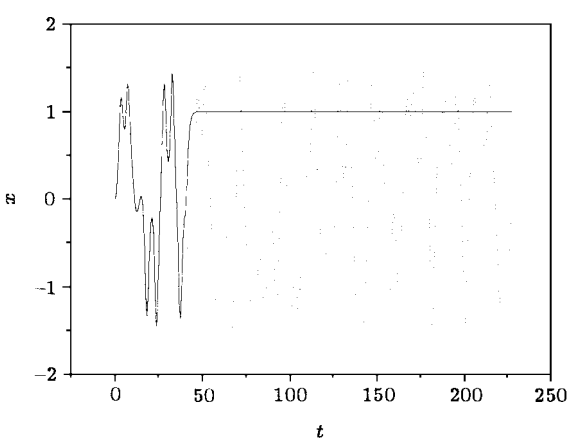


Fig.3. Control of oscillator (22) to the fixed point.

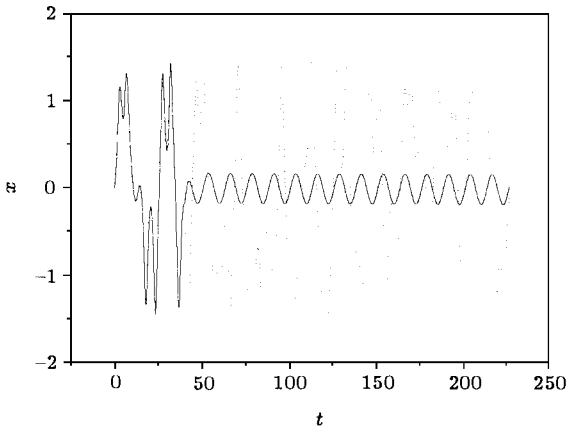


Fig.4. Control of oscillator (22) to the period-two motion.

The forced van der Pol oscillator with a controllable parameter[30]

$$\ddot{x} = -5(x^2 - 1)\dot{x} - x + 5\cos 2.466t + u \quad (26)$$

is taken as another example.

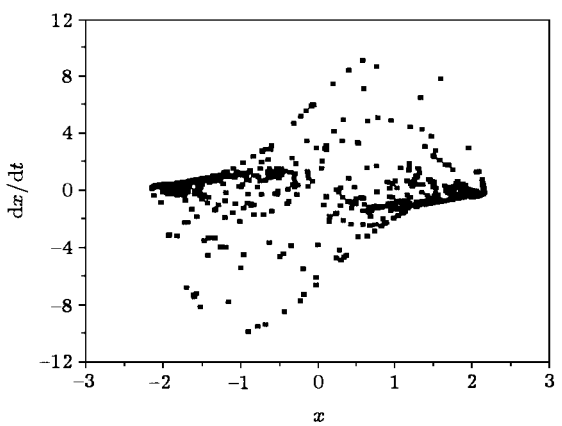


Fig.5. The Poincaré map of oscillator (26) without control.

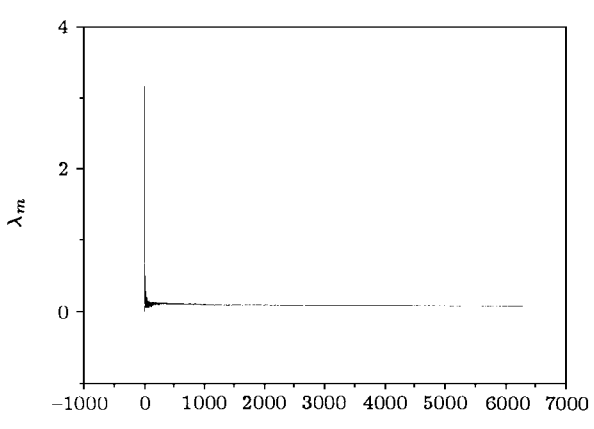


Fig.6. The largest Lyapunov exponent of oscillator (26) without control.

If no control is applied, chaos occurs in oscillator (26). The Poincaré map of the uncontrolled case is shown in Fig.5. The corresponding Lyapunov exponents are 0.096, 0.000 and –6.820, and the largest Lyapunov exponent varying with time is depicted in Fig.6.

Letting

$$f_{\rm P}(x,\dot{x},t) = -5(x^2-1)\dot{x} - x + 5\cos 2.466t, \quad (27)$$

then for given control goal $g(t)$ chosen as a fixed point and a period-four motion successively

$$g(t) = 0, \quad g(t) = \sin 0.6165t, \quad (28)$$

Eqs.(21), (8) and (9) give OPNCL control law of system (26)

$$u = S(t)[\ddot{g} - f_{\rm D}(g,\dot{g},t) + (\alpha - 1 - 5\dot{g}x - 5g\dot{x})(g-x) + (\beta + 5 + 5gx - 5g^2 - 5x^2)(\dot{g}-\dot{x})]. \quad (29)$$

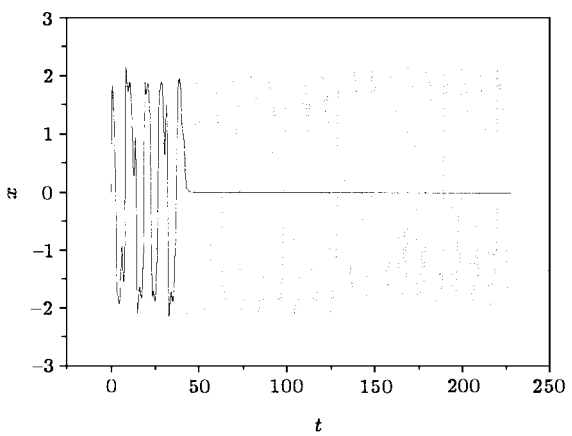


Fig.7. Control of oscillator (26) to the fixed point.

We start control at t_1=40.0, and we choose α=4.0 and β=2.8. The results are, respectively, shown in Figs.7 and 8, in which the solid and dotted lines correspond to the time histories of the controlled and uncontrolled systems, respectively.

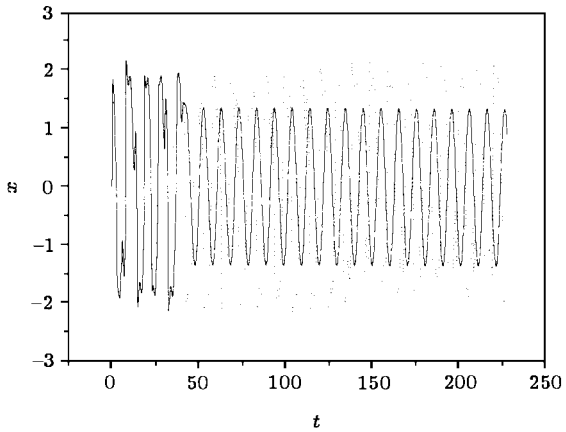


Fig.8. Control of oscillator (26) to the period-four motion.

4. Conclusions

Here, the OPNCL control law (7) or (20) for chaos in nonlinear oscillations is presented. It is proven that the basins of entrainment are the whole phase space for systems whose right-hand sides are polynomials. The applications of the OPNCL method in the forced Duffing oscillator and the forced van der Pol oscillator are discussed, demonstrating the effectiveness of the method.

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