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First quantum correction to entropy of Vaidya–Bonner black holes due to arbitrary spin fields*

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Using the improved brick-wall model, we have calculated the first quantum correction to the entropy of non-static black holes, Vaidya–Bonner black holes, due to the gravitational, electro-magnetic and neutrino fields. The result shows that both bosonic entropy and fermionic entropy are exactly proportional to the area of the event horizon. Thus, the entropy-area law still holds in such a non-static case.

Keywords: entropy, non-static black hole, membrane model, spin
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1. Introduction

Ever since Bekenstein and Hawking proposed, in the 1970s, that the static or stationary black hole entropy is proportional to the area of the event horizon, the origin of the black hole entropy remains a fascinating problem. For computing the black hole entropy, 't Hooft proposed a brick-wall model in which the black hole entropy is identified with the entropy of the quantum fields surrounding the black hole itself.[1] Taking the scalar field in Schwarzschild space–time for example, he obtained the same conclusion. After that, a series of work on calculating the black hole entropy appeared via this model. Researchers considered all kinds of fields in variant static or stationary space–times and all obtain the same result.[2−26,29,30]

However, researchers mainly study the entropy of static or stationary black holes. As for the non-static black hole, we do not know whether the black hole entropy is proportional to the area of the event horizon. Li and Zhao[25] have studied the scalar field (spin $s = 0$) entropy of Vaidya black holes.

In this paper, we give the calculations of the first quantum correction to the entropy of non-static Vaidya–Bonner black holes due to the neutrino field (spin $s = 1/2$), electro-magnetic field (spin $s = 1$) and gravitational field (spin $s = 2$). The result shows that both bosonic entropy and fermionic entropy are exactly proportional to the area of the event horizon. The result is similar to the static black hole case.[8]

In the brick-wall method, due to technical difficulties, one has to make some suitable approximation. In fact, the leading term of entropy in the brick-wall method comes from the contribution of the field near the horizon. As is well known, Hawking radiation comes from the vacuum fluctuation in the vicinity of the horizon; we think the black hole entropy does too. According to this idea, we have improved the brick-wall method.[20,25,26] For simplicity in mathematics, we use the improved brick-wall method in which the black hole entropy is identified with the entropy of a layer quantum field in the vicinity of the event horizon.

2. Entropy of the Vaidya–Bonner black hole

The metric of the Vaidya–Bonner black hole reads[27]

$$\mathrm{d}s^2 = \left(1 - \frac{2M(v)}{r} + \frac{Q(v)^2}{r^2}\right)\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right), \quad (1)$$

where $M(v)$ and $Q(v)$ are the mass and the charge of black hole, respectively.

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Choosing the null tetrad below, we have

$$\begin{aligned} l_\mu =&(1,0,0,0)\,,\\ n_\mu =&\left[\frac{1}{2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right),-1,0,0\right],\\ m_\mu =&-\frac{r}{\sqrt{2}}(0,0,1,\mathrm{i}\sin\theta)\,,\\ \overline{m}_\mu =&-\frac{r}{\sqrt{2}}(0,0,1,-\mathrm{i}\sin\theta)\,. \end{aligned} \tag{2}$$

The corresponding covariant null tetrad is

$$\begin{aligned} l^\mu =&(0,-1,0,0)\,,\\ n^\mu =&\left[1,\frac{1}{2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right),0,0\right],\\ m^\mu =&\frac{1}{\sqrt{2}r}\left(0,0,1,\frac{\mathrm{i}}{\sin\theta}\right),\\ \overline{m}^\mu =&\frac{1}{\sqrt{2}r}\left(0,0,1,-\frac{\mathrm{i}}{\sin\theta}\right). \end{aligned} \tag{3}$$

Then non-vanishing spin coefficients can be obtained as

$$\rho=\frac{1}{r},\quad \mu=\frac{1}{2r}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right),$$

$$\gamma=-\frac{Mr-Q^2}{2r^3},\quad \alpha=-\beta=-\frac{1}{2\sqrt{2}r}\mathrm{ctg}\theta. \tag{4}$$

Only one component of the Weyl tensor is non-vanishing, i.e.

$$\Psi_2=-\frac{Mr-Q^2}{r^4}. \tag{5}$$

Equations (4) and (5) show that the Vaidya–Bonner metric is of the Petrov-type D. Using the result of Teukolsky and Press,[28] the field equations of spin $s=1/2,1$, and 2 for the source-free case can be combined into

$$\begin{aligned} &[D-(2s+1)\rho][\Delta-2s\gamma+\mu]\,\Phi_{+s}-[\delta+(2s-2)\alpha]\\ &\times[\overline{\delta}-2s\alpha]\,\Phi_{+s}+(2s-1)(s-1)\,\Psi_2\,\Phi_{+s}=0,\\ &[\Delta+(2s-2)\gamma+(2s+1)\mu][D-\rho]\,\Phi_{-s}\\ &-[\overline{\delta}+(2s-2)\alpha][\delta-2s\alpha]\,\Phi_{-s}\\ &+(2s-1)(s-1)\,\Psi_2\,\Phi_{-s}=0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} D=&-\frac{\partial}{\partial r},\\ \Delta=&\frac{\partial}{\partial v}+\frac{1}{2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\frac{\partial}{\partial r},\\ \delta=&\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right),\\ \overline{\delta}=&\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right). \end{aligned} \tag{7}$$

In Eqs.(6) the first equation is for spin state $+s$, while the other is for $-s$.

Considering the symmetry of space–time, we set

$$\Phi_{\pm s}=R_{\pm s}(v,r)\,\mathrm{Y}_{lm}(\theta,\varphi)\,. \tag{8}$$

Substituting Eqs.(4), (5), (7) and (8) into Eqs.(6), then the radial and angular equations are obtained

$$\begin{aligned} &[D-(2s+1)\rho]\,[\Delta-2s\gamma+\mu]\,R_{+s}\\ &-(2s-1)(s-1)\,\Psi_2R_{+s}=-\frac{\lambda^2}{2r^2}R_{+s},\\ &[\Delta+(2s-2)\gamma+(2s+1)\mu]\,[D-\rho]\,R_{-s}\\ &-(2s-1)(s-1)\,\Psi_2R_{-s}=-\frac{\lambda^2}{2r^2}R_{-s}, \end{aligned} \tag{9}$$

$$\begin{aligned} &\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta)+\frac{1}{\sin^2\theta}\partial_\varphi^2+\frac{2\mathrm{i}s\cos\theta}{\sin^2\theta}\partial_\varphi\right]\mathrm{Y}_{lm}(\theta,\varphi)\\ &-\left[s^2\mathrm{ctg}^2\theta+s-\lambda^2\right]\mathrm{Y}_{lm}(\theta,\varphi)=0, \end{aligned} \tag{10}$$

where $\lambda^2=(l-s)(l+s+1)$ is the separation constant.

Following Ref.[25], we make coordinate transformations below

$$R=r-\dot{r}_+,\quad V=v-v_0, \tag{11}$$

where $\dot{r}_+=\mathrm{d}r_+/\mathrm{d}v$, r_+ is the radius of the event horizon. Then we have

$$\frac{\partial}{\partial r}=\frac{\partial}{\partial R},\quad \frac{\partial}{\partial v}=-\dot{r}_+\frac{\partial}{\partial R}+\frac{\partial}{\partial V}. \tag{12}$$

Substituting Eqs.(4), (5), (7) and (12) into Eqs.(9) and letting

$$R_{\pm s}=\mathrm{e}^{\mathrm{i}(Z-\omega V)}, \tag{13}$$

then the wavenumbers of incoming($k_-^{\pm s}$) and outgoing ($k_+^{\pm s}$) waves are obtained

$$\begin{aligned} k_\pm^{-s}\equiv&\frac{\mathrm{d}Z}{\mathrm{d}R}\\ =&\frac{\omega\pm\sqrt{\omega^2-\left(1-\dfrac{2M}{r}+\dfrac{Q^2}{r^2}-2\dot{r}_+\right)\dfrac{\lambda^2-A(v,r)}{r^2}}}{1-\dfrac{2M}{r}+\dfrac{Q^2}{r^2}-2\dot{r}_+}, \end{aligned} \tag{14}$$

$$\begin{aligned} k_\pm^{+s}\equiv&\frac{\mathrm{d}Z}{\mathrm{d}R}\\ =&\frac{\omega\pm\sqrt{\omega^2-\left(1-\dfrac{2M}{r}+\dfrac{Q^2}{r^2}-2\dot{r}_+\right)\dfrac{\lambda^2-2s}{r^2}}}{1-\dfrac{2M}{r}+\dfrac{Q^2}{r^2}-2\dot{r}_+}, \end{aligned} \tag{15}$$

where

$$A(v,r) = -4s^2\left(\frac{M}{r}-\frac{Q^2}{r^2}\right) + 2s\left(\frac{1}{r^2}-\frac{Q^2}{r^2}\right)+2\dot{r}_+. \quad (16)$$

Assuming that the quantum field which surrounds the black hole has the same temperature as the black hole at any time, we can take it for granted that the quantum field is in equilibrium with the hole at any time. Using quasi-stationary approximation at some time, we obtain the number of radial modes n

$$2n\pi = \int_{\varepsilon}^{2\varepsilon} k_+^{-s}\mathrm{d}R + \int_{2\varepsilon}^{\varepsilon} k_-^{-s}\mathrm{d}R, \quad (17)$$

where $\varepsilon \ll r_+$. Equation (17) can be rewritten as

$$n\pi = \int_{r_++\varepsilon}^{r_++2\varepsilon} \mathrm{d}r\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}-2\dot{r}_+\right)^{-1} \times\left[\omega^2-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}-2\dot{r}_+\right)\times\frac{\lambda^2-A(v,r)}{r^2}\right]^{1/2}. \quad (18)$$

Then the free energy is obtained as

$$\begin{aligned}\beta F_1 &= \mp\sum_{n,l,m}\ln\left(1\pm\mathrm{e}^{-\beta\omega}\right)\\ &\approx \mp\int\mathrm{d}l\,(2l+1)\int\mathrm{d}n\ln\left(1\pm\mathrm{e}^{-\beta\omega}\right)\\ &= -\int\mathrm{d}l\,(2l+1)\int\mathrm{d}\left(\beta\omega\right)\left(\mathrm{e}^{\beta\omega}\pm1\right)^{-1}n, \quad (19)\end{aligned}$$

that is

$$F_{1\mathrm{fermions}} = -\frac{7}{180}\frac{\pi^3}{\beta^3}\frac{r_+^6}{\left(r_+-r_-\right)^2}\frac{\varepsilon}{\eta^2}, \quad (20)$$

$$F_{1\mathrm{bosons}} = -\frac{2}{45}\frac{\pi^3}{\beta^3}\frac{r_+^6}{\left(r_+-r_-\right)^2}\frac{\varepsilon}{\eta^2}. \quad (21)$$

It should be noted that we have used the median theorem in the integration over r, hence $\varepsilon<\eta<2\varepsilon$. We have also used the event horizon equation $1-\frac{2M}{r}+\frac{Q^2}{r^2}-2\dot{r}=0$, i.e. $\frac{1}{r^2}(r-r_+)(r-r_-)=0$ and the relation $\left[\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}-2\dot{r}\right)A\right]_{r_++\eta}\approx 0$. r_- is the radius of the inner horizon.

From the relation between entropy and free energy

$$S = \beta^2\frac{\partial F}{\partial\beta}, \quad (22)$$

we obtain the entropy of one component of the wavefunction

$$S_{1\mathrm{fermions}} = \frac{7\pi^3}{45}\frac{1}{\left(r_+-r_-\right)^2}\frac{r_+^6}{\beta^3}\frac{h}{\eta^2}, \quad (23)$$

$$S_{1\mathrm{bosons}} = \frac{8\pi^3}{45}\frac{1}{\left(r_+-r_-\right)^2}\frac{r_+^6}{\beta^3}\frac{h}{\eta^2}. \quad (24)$$

The temperature of the Vaidya–Bonner black hole can be written as

$$T = \frac{1}{\beta} = \frac{r_+-r_-}{4\pi r_+^2}, \quad (25)$$

where ε and η are of the same order, i.e. $\varepsilon\sim\eta$. Thus we have $\frac{\varepsilon}{\eta^2}\sim\frac{1}{\varepsilon}$. According to 't Hooft's proposal we choose $\frac{\varepsilon}{\eta^2}\sim\frac{1}{\varepsilon}=90\beta$. Substituting Eq.(25) into Eqs.(23) and (24) and setting $\frac{\varepsilon}{\eta^2}=90\beta$, we have

$$S_{1\mathrm{fermions}} = \frac{7}{8}\cdot\frac{1}{4}A, \quad (26)$$

$$S_{1\mathrm{bosons}} = \frac{1}{4}A, \quad (27)$$

where $A=4\pi r_+^2$, A is the area of the event horizon.

It can be shown that the entropy of the R_{+s} component is the same as that of the R_{-s} component. The sum of the entropy is obtained as

$$S_{\mathrm{fermions}} = 2S_{1\mathrm{fermions}} = \frac{7}{4}\cdot\frac{1}{4}A, \quad (28)$$

$$S_{\mathrm{bosons}} = 2S_{1\mathrm{bosons}} = 2\cdot\frac{1}{4}A. \quad (29)$$

Equations (28) and (29) tell us that the first quantum correction to the entropy of the non-static Vaidya–Bonner black hole is exactly proportional to the area of its event horizon. Thus, the relation between the entropy and the event horizon in the static or stationary cases still holds in such a non-static case. Our result is similar to that in Ref.[8].

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