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Vacuum polarization of scalar electrodynamics in flat space–time with nontrivial topology*

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The one-loop vacuum polarization is calculated for scalar electrodynamics in a flat space–time with the topology $S^1 \times R^3$. The effect of vacuum polarization upon photon propagation is considered. It is found that photon propagation becomes anisotropic with some photon modes acquiring a positive topological mass and thus travelling at a subluminal speed.

Keywords: vacuum polarization, space–time with nontrivial topology, photon propagation
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1. Introduction

It is well known that physics is sensitive to both the local and global structure of space–time. While local space–time curvature can affect the field equations of the theory, the global properties of the space–time, even if it is locally flat, can give rise to many novel effects, such as the Casimir effect,[1] and the light-cone fluctuation effect when gravity is quantized.[2,3] In investigating effects in quantum field theories arising from the nontrivial topology of the flat space–time, Ford[4] has examined the effects of one-loop vacuum polarization of spinor electrodynamics in a nonsimply connected space–time which has a topology, $S^1 \times R^3$ (i.e. a Minskowski space with one spatial dimension compactified in a circle). It was found that some photon modes could develop a mass at the one-loop level as a result of the nontrivial topology. This phenomenon is called topological mass generation and has been studied extensively in the literature. In this respect, Ford[4] and Yoshimura[5] have shown, by computing the scalar particle self-energy in a $\lambda\phi^4$ self-interacting massless scalar field theory, that the massless scalar particles could also develop a topological mass at the one-loop level. Toms[6] and Yoshimura[7] have calculated the electromagnetic self-energy for massless charged scalar fields. The topological mass generation has also been discussed in the context of compact extra dimensions.[8]

It is interesting to note that the topological masses, which some photon modes acquire due to the vacuum polarization resulting from the creation and the subsequent annihilation of virtual spinor pairs in the nonsimply connected space–time, are imaginary.[4] Thus these modes are speeded up and would appear to travel at a superluminal speed. However, it should be pointed out that the photon propagation can also be influenced by the vacuum polarization effects produced by the creation and the subsequent annihilation of virtual charged scalar pairs. So, it remains interesting to examine the vacuum polarization of scalar electrodynamics in the $S^1 \times R^3$ space–time and to see what effects it will have on photon propagation. We discuss this in this paper, and we would like to note that this study could be of significance to theories with compactified extra spatial dimensions, which have attracted a great deal of attention in recent years.

2. Vacuum polarization

In this section, the one-loop vacuum polarization tensor of scalar electrodynamics is calculated to order e^2 in $S^1 \times R^3$ space–time. We assume that x^1 is compactified with a periodicity length L. The Lagrangian of scalar electrodynamics here is the same as that in Minkowski space–time:

$$\mathcal{L} = \frac{\partial\phi^*}{\partial x_\mu}\frac{\partial\phi}{\partial x^\mu} - m^2\phi^*\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mathrm{i}eA^\mu\phi^*(\overrightarrow{\partial_\mu} - \overleftarrow{\partial_\mu})\phi + e^2A_\mu\phi^*\phi A^\mu \,. \quad (1)$$

Here e is the bare charge and A_μ and ϕ are required to satisfy the periodic boundary condition in the x^1 direction. From Ref.[10] we obtain the one-loop vacuum

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polarization tensor

$$\begin{aligned}&\Pi_{\mu\nu}(q)\\ =&\frac{e^2}{L(2\pi)^3}\sum_{n=-\infty}^{\infty}\int \mathrm{d}^3p\\ &\times\left\{\frac{(2p_\mu-q_\mu)(2p_\nu-q_\nu)}{(p^2-m^2+\mathrm{i}\varepsilon)[(p-q)^2-m^2+\mathrm{i}\varepsilon]}\right.\\ &\left.-g_{\mu\nu}\frac{1}{p^2-m^2+\mathrm{i}\varepsilon}-g_{\mu\nu}\frac{1}{(p-q)^2-m^2+\mathrm{i}\varepsilon}\right\},\end{aligned}\tag{2}$$

where L is the period, $\mathrm{d}^3p=\mathrm{d}p^0\mathrm{d}p^2\mathrm{d}p^3$ and $p^1=2\pi nL^{-1}$, $n=0,\pm1,\pm2,....$

Now we use dimensional regularization to isolate the divergences in $\Pi_{\mu\nu}$.[9] Let us now generalize an $(N+1)$-dimensional space–time in which one spatial direction is compactified. To keep $\Pi_{\mu\nu}$ dimensionless in $(N+1)$ dimensions, we need to introduce an arbitrary renormalization length parameter l_R. So

$$\Pi^{(\mathrm{R})}_{\mu\nu}(q)=\frac{e^2l_{\mathrm{R}}^{N-3}}{L(2\pi)^N}\sum_{n=-\infty}^{+\infty}\int \mathrm{d}^Np\,\frac{(2p_\mu-q_\mu)(2p_\nu-q_\nu)-g_{\mu\nu}[(p-q)^2+p^2-2m^2]}{(p^2-m^2+\mathrm{i}\varepsilon)[(p-q)^2-m^2+\mathrm{i}\varepsilon]}\,.\tag{3}$$

Combining the tensor with a Feynman parameter integration

$$\begin{aligned}\frac{1}{(p^2-m^2)[(p-q)^2-m^2]}=&\int_0^1\mathrm{d}x\,\{(1-x)(p^2-m^2)+x[(p-q)^2-m^2]\}^{-2}\\ =&\int_0^1\mathrm{d}x\,(p^2+2p\cdot k+H)^{-\alpha}\,|_{\alpha=2,H=q^2x-m^2,k=-qx}\,,\end{aligned}\tag{4}$$

we obtain

$$\begin{aligned}\Pi^{(\mathrm{R})}_{\mu\nu}=&\frac{e^2l_{\mathrm{R}}^{(N-3)}}{L(2\pi)^N}\sum_n\int\mathrm{d}^Np\int_0^1\mathrm{d}x\,\frac{[(2p_\mu-q_\mu)(2p_\nu-q_\nu)-g_{\mu\nu}(2p^2-2p\cdot q+q^2-2m^2)]}{(p^2+2p\cdot k+H)^\alpha}\\ =&\frac{e^2l_{\mathrm{R}}^{(N-3)}}{L(2\pi)^N}\int_0^1\mathrm{d}x\,[4I_{\mu\nu}-2q_\mu I_\nu-2q_\nu I_\mu+q_\mu q_\nu I-2g_{\mu\nu}I^\rho_\rho\\ &+2g_{\mu\nu}q^\rho I_\rho-g_{\mu\nu}q^2I+g_{\mu\nu}2m^2I]\,,\end{aligned}\tag{5}$$

where

$$\begin{aligned}I(N,\alpha;q,H)=&\sum_n\int\mathrm{d}^Np\frac{1}{(p^2+2p\cdot k+H+\mathrm{i}\varepsilon)^\alpha}\\ =&\mathrm{i}(-1)^{-\alpha}\pi^{N/2}\frac{\Gamma\left(\alpha-\frac{N}{2}\right)}{\Gamma(\alpha)}\left(\frac{2\pi}{L}\right)^{N-2\alpha}F\left(\alpha-\frac{N}{2};a,b\right)\,,\end{aligned}\tag{6}$$

$$\begin{aligned}I_\mu(N,\alpha;q,H)=&\sum_n\int\mathrm{d}^Np\,\frac{p_\mu}{(p^2+2p\cdot k+H+\mathrm{i}\varepsilon)^\alpha}\\ =&\mathrm{i}(-1)^{-\alpha}\pi^{N/2}\frac{\Gamma\left(\alpha-1-\frac{N}{2}\right)}{2\Gamma(\alpha)}\left(\frac{2\pi}{L}\right)^{N-2\alpha+1}\\ &\times\left[(k^2-H)^{-1/2}k_\mu\frac{\partial}{\partial a}\,F\left(\alpha-1-\frac{N}{2};a,b\right)+n_\mu\frac{\partial}{\partial b}F\left(\alpha-1-\frac{N}{2};a,b\right)\right]\,,\end{aligned}\tag{7}$$

and

$$\begin{aligned}&I_{\mu\nu}(N,\alpha;q,H)\\ =&\sum_n\int\mathrm{d}^Np\,\frac{p_\mu p_\nu}{(p^2+2p\cdot k+H+\mathrm{i}\varepsilon)^\alpha}\\ =&\mathrm{i}(-1)^{-\alpha}\pi^{N/2}\frac{\Gamma\left(\alpha-2-\frac{N}{2}\right)}{4\Gamma(\alpha)}\left(\frac{2\pi}{L}\right)^{N-2\alpha+2}\left[g_{\mu\nu}a^{-1}\frac{\partial}{\partial a}+\left(\frac{L}{2\pi}\right)^2a^{-2}k_\mu k_\nu\left(\frac{\partial^2}{\partial a^2}-a^{-1}\frac{\partial}{\partial a}\right)\right.\\ &\left.+\frac{L}{2\pi}a^{-1}(k_\mu n_\nu+k_\nu n_\mu)\frac{\partial^2}{\partial a\partial b}+n_\mu n_\nu\frac{\partial^2}{\partial b^2}\right]F\left(\alpha-2-\frac{N}{2};a,b\right)\Big|_{a=(k^2-H)^{1/2}L/2\pi,b=n\cdot kL/2\pi}\,.\end{aligned}\tag{8}$$

Here $n^\mu = (0, 1, 0, ..., 0)$ is a unit vector which points to the direction of periodicity and $F(\lambda; a, b)$ is defined as

$$F(\lambda; a, b) = \sum_{n=-\infty}^{\infty} [(n+b)^2 + a^2]^{1/2} . \qquad (9)$$

For real a, which corresponds to the requirement that $m^2 \geq q^2$, and $\mathrm{Re}\lambda > 1/2$,[4]

$$\begin{aligned} & F(\lambda, a, b) \\ =& \pi^{1/2} a^{1-2\lambda} \frac{\Gamma\left(\lambda - \frac{1}{2}\right)}{\Gamma(\lambda)} + 4\sin(\lambda\pi) f_\lambda(a, b) , \end{aligned} \qquad (10)$$

where

$$f_\lambda(a, b) = \mathrm{Re} \int_a^\infty \mathrm{d}u \ \frac{(u^2 - a^2)^{-\lambda}}{\exp[2\pi(u + \mathrm{i}b)] - 1} . \qquad (11)$$

Using Eq.(10) we can write the vacuum polarization tensor as

$$\Pi_{\mu\nu}^{(\mathrm{R})}(q) = \Pi_{\mu\nu}^{(0)}(q) + \Pi_{\mu\nu}^{(1)}(q) . \qquad (12)$$

Here $\Pi_{\mu\nu}^{(0)}(q)$ is the contribution arising from the first term in Eq.(10) and is the usual Minkowski space–time vacuum polarization tensor of scalar electrodynamics to order e^2 calculated in dimensional regularization. In the limit $L \to \infty$, we can replace the sums in $\Pi_{\mu\nu}$ by an integral to obtain

$$\begin{aligned} & \Pi_{\mu\nu}^{(0)}(q) \\ =& \mathrm{i} \frac{2e^2 l_{\mathrm{R}}^{N-3}}{(2\pi)^{N+1}} \Gamma\left(\frac{3}{2} - \frac{N}{2}\right) \int_0^1 \mathrm{d}x \ \left(- x^2 + x - \frac{1}{2}\right) \\ & \times [q^2 x(x-1) + m^2]^{N/2-3/2} (g_{\mu\nu} q^2 - q_\mu q_\nu) . \end{aligned} \qquad (13)$$

Apparently $\Pi_{\mu\nu}^{(0)}$ is divergent when $N = 3$, but the pole terms can be absorbed by charge renormalization. So we obtain the finite Minkowski space–time vacuum polarization tensor of scalar electrodynamics

$$\begin{aligned} \Pi_{\mu\nu}^{(0)}(q)_{\mathrm{finite}} =& \mathrm{i} e_B^2 \pi^{-1} (q_\mu q_\nu - g_{\mu\nu} q^2) \\ & \times \int_0^1 \mathrm{d}x \ \left[x(1-x) - \frac{1}{2} \right] \\ & \times \ln \left[1 - \frac{q^2}{m^2} x(1-x) \right] . \end{aligned} \qquad (14)$$

Here e_B is the renormalized charge. $\Pi_{\mu\nu}^{(1)}(q)$ is the contribution arising from the second term in Eq.(10), which is finite at $N = 3$, and can be expressed as

$$\begin{aligned} & \Pi_{\mu\nu}^{(1)}(N=3) \\ =& \mathrm{i} e^2 \pi^{-2} \int_0^1 \mathrm{d}x \ \Big\{ (q_\mu q_\nu - g_{\mu\nu} q^2) \\ & \times \left[x(x-1) + \frac{1}{2} \right] f_{1/2} + \pi L^{-1} (2x-1) \\ & \times (q_\mu n_\nu + n_\mu q_\nu - q{\cdot}n g_{\mu\nu}) \dot{f}_{-1/2} \\ & - n_\mu n_\nu [x(x-1) q^2 f_{1/2} \\ & + m^2 f_{1/2} + 8\pi^2 L^{-2} f_{-1/2}] \Big\} . \end{aligned} \qquad (15)$$

Here $\dot{f}_\lambda = (\partial/\partial b) f_\lambda(a, b)$. In the course of the calculation we have used the following relations

$$\left(\frac{\partial}{\partial a} \right) f_\lambda(a, b) = 2a\lambda f_{\lambda+1}(a, b) , \qquad (16)$$

$$\begin{aligned} & \left(\frac{\partial^2}{\partial b^2} \right) f_\lambda(a, b) \\ =& -2\lambda(2\lambda+1) f_{\lambda+1}(a, b) - 4\lambda(\lambda+1) a^2 f_{\lambda+2}(a, b) . \end{aligned} \qquad (17)$$

$\Pi_{\mu\nu}^{(1)}(q)$ clearly depends on the compactification length L, and therefore it reflects the contribution due to the nontrivial topology. This result is quite different physically from that obtained in spinor electrodynamics in that the overall sign is positive rather than negative. The consequence of this fact will be discussed later. Now we want to address the gauge invariance issue. At first sight, $\Pi_{\mu\nu}^{(1)}(q)$ does not seem to obey the condition of gauge invariance:

$$q^\mu \Pi_{\mu\nu}(q) = 0 . \qquad (18)$$

However, if we note that[4]

$$\begin{aligned} & \int_0^1 \mathrm{d}x \ \{ \pi L^{-1} (1-2x) q^2 \dot{f}_{-1/2} \\ & + q{\cdot}n [x(x-1) q^2 f_{1/2} \\ & + m^2 f_{1/2} + 8\pi^2 L^{-2} f_{-1/2}] \} = 0 , \end{aligned} \qquad (19)$$

we can rewrite Eq.(15) as

$$\begin{aligned} \Pi_{\mu\nu}^{(1)} =& \mathrm{i} e^2 \pi^{-2} \int_0^1 \mathrm{d}x \ \Big\{ (q_\mu q_\nu - g_{\mu\nu} q^2) \\ & \times \left[x(x-1) + \frac{1}{2} \right] f_{1/2} \\ & + \pi L^{-1} (2x-1) [q_\mu n_\nu + n_\mu q_\nu \\ & - q{\cdot}n g_{\mu\nu} - q^2 n_\mu n_\nu (q \cdot n)^{-1}] \dot{f}_{-1/2} \Big\} . \end{aligned} \qquad (20)$$

In this form the gauge invariance is manifested.

3. Topological mass

Now let us examine the effect of vacuum polarization of scalar electrodynamics upon the photon propagator. The propagator to order e^2 can be expressed as

$$D_{\mu\nu}(q) = -\frac{g_{\mu\nu}}{q^2} + \mathrm{i}\frac{1}{q^2}\Pi_{\mu\nu}(q)\frac{1}{q^2} \,. \tag{21}$$

Following Ref.[4], we consider an important special case, i.e. the real photons propagating in a direction perpendicular to n_μ, then we have $q^2 = q\cdot n = 0$. Thus we obtain

$$\Pi^{(1)}_{\mu\nu}(q) = \frac{1}{3}\mathrm{i}e^2\pi^{-2}q_\mu q_\nu f_{1/2}(\xi,0) - \mathrm{i}Bn_\mu n_\nu \mid_{\xi=Lm/2\pi} \,, \tag{22}$$

where

$$B = 4e^2L^{-2}[2f_{-1/2}(\xi,0) + \xi^2 f_{1/2}(\xi,0)] \,. \tag{23}$$

Here we have used the fact that $\dot{f}_\lambda(a,0) = 0$. In particular

$$\Pi^{(1)}_{\mu\nu}(0) = -\mathrm{i}Bn_\mu n_\nu \,. \tag{24}$$

Thus, if we set $\Pi_{\mu\nu} = -\mathrm{i}Bn_\mu n_\nu$ (B is given by Eq.(23)) and let both the motion direction (q^ρ) and polarization direction (ϵ^μ) of a plane wave be perpendicular to n_μ, then

$$\epsilon^\mu_\perp D_{\mu\nu}\epsilon^\nu_\perp = \frac{1}{q^2} \,. \tag{25}$$

But if we let ϵ^μ be parallel to n^ν and q^ρ be as before, then

$$\epsilon^\mu_\parallel D_{\mu\nu}\epsilon^\nu_\parallel = \frac{1}{q^2} + \frac{B}{(q^2)^2} = \frac{1}{q^2-B} + O(e^4) \,. \tag{26}$$

So if the polarization vector is parallel to n^ν, a plane wave travelling in a perpendicular direction propagates as though the field had a mass $\mu = \sqrt{B}$, and clearly the mass is gauge independent. If the polarization vector is perpendicular, the mode propagates as a massless field. The mass arises from the topological structure. If $L \to \infty$, the mass vanishes as expected. Note that other terms in $\Pi^{(1)}_{\mu\nu}$ have no effect on the propagation of photons for which $q^2 = q\cdot n = 0$, since terms proportional to $q^2 g_{\mu\nu}$ may be absorbed by a charge renormalization and other terms vanish when contracted into polarization vectors. It can be shown by a similar analysis that, for photon modes for which $q\cdot n \neq 0$, both polarization states acquire an effective mass of

$$\mu'^2 = 4e^2L^{-2}f_{1/2}(\xi,0) \,. \tag{27}$$

Let us examine the limits of large and small L. Using the following asymptotic forms for $f_\lambda(a,b)$

$$f_\lambda(a,b) \sim 1/2\pi^{\lambda-1}a^{-\lambda}\Gamma(1-\lambda)\cos 2\pi b\mathrm{e}^{-2\pi a}, \quad a\to\infty, \tag{28}$$

and

$$f_\lambda(0,0) \sim (2\pi)^{2\lambda-1}\Gamma(1-2\lambda)\zeta(1-2\lambda) \,, \quad a,b\to 0 \,, \quad \mathrm{Re}\lambda < 0, \tag{29}$$

we find

$$\mu^2 \sim e^2 m^{3/2}\pi^{-3/2}L^{-1/2}\mathrm{e}^{-mL},$$

$$\mu'^2 \sim e^2 m^{1/2}\pi^{-3/2}L^{-3/2}\mathrm{e}^{-mL}, \quad L \geq m^{-1} \,, \tag{30}$$

and

$$\mu^2 \sim e^2/3L^2 \,, \quad \mu'^2 \sim e^2/6L^2 \,, \quad L \leq m^{-1} \,. \tag{31}$$

The important thing to notice here is that the topological mass is real and positive and thus the photon modes in question will travel at a subluminal speed. This is in sharp contrast to the effects on photon propagation by the vacuum polarization in spinor electrodynamics where the topological mass is imaginary and the photon modes propagate at a superluminal speed.[4]

4. Conclusion

In summary, we have calculated the one-loop vacuum polarization of scalar electrodynamics in a flat space–time with the topology $S^1 \times R^3$ and we have examined its influence on photon propagation. We find that photons travelling along the compactification direction propagate as if they had a positive mass and therefore travel at a subluminal speed. But, for photons moving perpendicular to the compactification direction, only modes with longitudinal polarization will acquire a mass and those with transverse polarization remain massless. Thus the photon propagation becomes anisotropic. It is interesting to note that the topological mass which some photon modes would develop as a result of the vacuum polarization in scalar electrodynamics is real and positive while that in spinor electrodynamics is imaginary. Thus, they have just the opposite effects on photon propagation, leading to slowing down or speeding up the propagation of some photon modes, respectively. To conclude, we would like to point out that our result could be of interest to theories with compactified extra spatial dimensions, which have attracted a great deal of attention in recent years.

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