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The non-local implementation of a CNOT gate and single-qubit rotation*

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We show how a CNOT gate and single-qubit rotation can be implemented non-locally. We also report on the quantitative relations between these quantum actions, entanglement and classical communication resources required in the implementation.

Keywords: non-local implementation, CNOT gate, single-qubit rotation, entangled state
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1. Introduction

A quantum computer[1−3] allows, in principle, the efficient solution of some problems that are intractable on a classical computer. However, the practical problems are daunting; these range from fundamental effects, such as decoherence and dissipation and experimental imperfections, to the nontrivial task of storing and isolating reliably a large number of qubits.[3,4−6] A possible solution would be the construction of a quantum computer not as a local device that contains all qubits in a single processor, but to build it from the outset as a multiprocessor device where each processor contains only a small number of qubits. Such a "distributed quantum computer"[7] can be viewed as a generalization of a quantum communication network in which each node can act as a sender or receiver and contains only a small number of qubits. Before one is able to consider the physical resource efficiency of a distributed quantum computer in general, it is first necessary to establish optimal implementations of quantum operations between qubits that are located in different nodes of the distributed quantum computer.

In the past, most research on quantum non-locality has been devoted to the issue of the non-locality of quantum states.[8−11] However, we feel that an equally important issue is that of non-locality of quantum operations. The basic operation in a quantum computer is unitary evolution. General single-qubit rotations together with a CNOT gate are sufficient to implement any multi-qubit unitary transformation.[12] This implies that the non-local implementations of a CNOT gate and single-qubit rotation are essential to the construction of general unitary transformation in a distributed quantum computer. In this paper, we investigate the non-local implementation of a quantum CNOT gate and single-qubit rotation.

Consider that Alice and Bob are localized in different space domains, and are able to perform any local operation. They also have additional resources, namely they share entangled states, and they are able to communicate classically.

In all the quantum circuits described in this paper, we use the usual convention that time proceeds from left to right, so that the leftmost gate operates first, etc. We write X, Y and Z instead of σ_x, σ_y and σ_z.

2. Non-local single-qubit rotation

In the following we construct a quantum circuit (Fig.1) which allows Bob to remotely "rotate" Alice's qubit.

For simplicity, we restrict our attention to the rotations of the form

$$R(\pm\theta) \equiv \exp(\pm \mathrm{i} Y\theta), \tag{1}$$

for an arbitrary angle $\theta \in (0, 2\pi]$, which corresponds to an arbitrary rotation around $\hat{y}$-axis.

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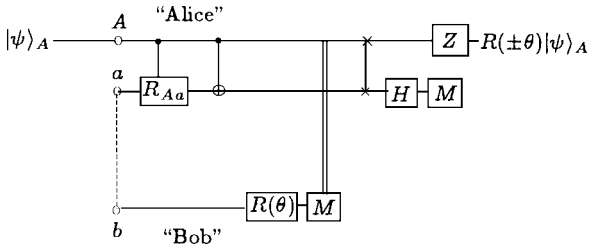


Fig.1. A quantum circuit for the non-local implementation of a single-qubit rotation $R(\theta)$. The dashed line connecting a and b means that they are entangled and classical communication is represented by a double line. The measurement (the box "M") is based on the computation.

Suppose Alice holds the qubits A and a, qubit A is in an arbitrary unknown state

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A, \tag{2}$$

and Bob holds the qubit b. The dashed line connecting a and b means that they share a maximally entangled state. So, the state of the whole system is

$$(\alpha|0\rangle_A + \beta|1\rangle_A)(|00\rangle_{ab} + |11\rangle_{ab}). \tag{3}$$

Step 1: Alice first performs a control-rotation operation R_{Aa} on her qubits A and a (with the control being qubit A)

$$R_{Ab} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|, \tag{4}$$

then a CNOT. Eq.(3) will be transformed into

$$(\alpha|00\rangle_{Aa} + \beta|11\rangle_{Aa})|0\rangle_b + (\alpha|01\rangle_{Aa} - \beta|10\rangle_{Aa})|1\rangle_b.$$

(Here, and in what follows, we leave out normalization factors for states.)

Step 2: Bob performs a rotation operation $R(\theta)$ on his qubit

$$\begin{aligned}&(\alpha|00\rangle_{Aa} + \beta|11\rangle_{Aa})(\cos\theta|0\rangle_b + \sin\beta|1\rangle_b)\\ &+ (\alpha|01\rangle_{Aa} - \beta|10\rangle_{Aa})(-\sin\theta|0\rangle_b + \cos\theta\langle 1|_b).\end{aligned} \tag{5}$$

Then, Bob performs a measurement on his qubit, and here the line corresponding to this qubit terminates. The result of the measurement is one classical bit (one c-bit) of information, which is communicated to Alice. If the result is $|0\rangle_b$, from Eq.(5), we have

$$(\alpha|00\rangle_{Aa} + \beta|11\rangle_{Aa})\cos\theta - (\alpha|01\rangle_{Aa} - \beta|10\rangle_{Aa})\sin\theta. \tag{6}$$

Step 3: A simple SWAP[13] operation on qubits A and a by Alice will transform Eq.(6) as follows

$$\begin{aligned}&(\alpha|00\rangle_{Aa} + \beta\langle 11|_{Aa})\cos\theta\\ &- (\alpha|10\rangle_{Aa} - \beta|01\rangle_{Aa})\sin\theta\\ =&\alpha|0\rangle_a|\phi\rangle_A + \beta|1\rangle_a|\phi'\rangle_A,\end{aligned} \tag{7}$$

where

$$|\phi\rangle_A = \cos\theta|0\rangle_A - \sin\theta|1\rangle_A,$$

$$|\phi'\rangle_A = \sin\theta|0\rangle_A + \cos\theta|1\rangle_A. \tag{8}$$

Now Alice performs a Hadamard transform on her qubit a, after which Eq.(7) becomes

$$\alpha(|0\rangle_a + |1\rangle_a)|\phi\rangle_A + \beta(|0\rangle_a - |1\rangle_a)|\phi'\rangle_A. \tag{9}$$

One can easily find from Eq.(9) that if Alice performs a measurement on her qubit a:

(1) for the measurement outcome $|0\rangle_a$, Eq.(9) will be

$$\begin{aligned}\alpha|\phi\rangle_A + \beta|\phi'\rangle_A =&\alpha(\cos\theta|0\rangle_A - \sin\theta|1\rangle_A)\\ &+ \beta(\sin\theta|0\rangle_A + \cos\theta|1\rangle_A)\\ =&R(-\theta)|\psi\rangle_A;\end{aligned} \tag{10}$$

(2) for the measurement outcome $|1\rangle_a$, Eq.(9) will be

$$\begin{aligned}\alpha|\phi\rangle_A - \beta|\phi'\rangle_A =&\alpha(\cos\theta|0\rangle_A - \sin\theta|1\rangle_A)\\ &+ \beta(-\sin\theta|0\rangle_A - \cos\theta|1\rangle_A)\\ =&ZR(\theta)|\psi\rangle_A.\end{aligned} \tag{11}$$

Equations (10) and (11) imply that, when qubit a is measured in state $|0\rangle_A$, Bob applies a rotation $R(-\theta)$ to a distant qubit A through his local operations. On the other hand, when qubit a is measured in state $|1\rangle_A$, the resulting state (11) of qubit A is the same as the state created by Alice directly performing a Pauli operator Z and then a rotation operation $R(\theta)$ on the initial state $|\psi\rangle_A$ of qubit A.

If Bob's measurement result is $|1\rangle_b$, it can be discussed in the same way.

The above discussion shows that, if Alice knows the result of Bob's measurement on his qubit, she can (by applying suitable local operation, such as phase and/or bit flips) put her qubit in a state which can be written compactly as $\exp(\pm iY\theta)|\psi\rangle_A$, where θ is the rotation angle chosen by Bob, and (depending on the result of her own measurement on qubit a) she will know whether the "+" or "–" sign applies in this expression.

In previous work, Huelga *et al*[14] have shown that if Bob wants to perform an arbitrary unitary remote operation on Alice's qubit, Bob and Alice need to share two e-bits, and also four c-bits are required. More recently[15] they have also considered the resources needed to perform restricted sets of operations, only one e-bit and two c-bits are needed. For the

case in our paper, one e-bit and one c-bit are enough for Bob to remotely "rotate" Alice's qubit.

3. Non-local CNOT gate

A more useful and important gate is the CNOT gate, which acts on two bits, a control and a target qubits, and flips the target whenever the control qubit is a $|1\rangle$ ($|a,b\rangle \rightarrow |a, a\oplus b\rangle$). In the following, we construct a quantum circuit (Fig.2) that performs the CNOT non-locally using one e-bit and two c-bits.

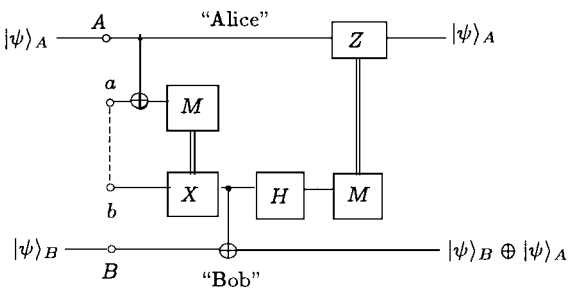


Fig.2. A quantum circuit for the non-local implementation of a CNOT gate.

Suppose Alice holds qubits A and a, Bob holds qubits B and b. a and b share a maximally entangled state $(|00\rangle_{ab} + |11\rangle_{ab})$. We show how to perform a CNOT operation on the state

$$|\psi\rangle_A|\psi\rangle_B = (\alpha|0\rangle_A + \beta|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B), \quad (12)$$

i.e. transform it to

$$\begin{aligned}&\mathrm{CNOT}|\psi\rangle_A|\psi\rangle_B\\ =&\alpha|0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B) + \beta|1\rangle_A(\gamma|1\rangle_B + \delta|0\rangle_B), \quad (13)\end{aligned}$$

where the CNOT gate has $|\psi\rangle_A$ as its control, and $|\psi\rangle_B$ as its target. Since the operation behaves linearly, the protocol performs the CNOT on any input state (i.e. even if the qubits are entangled with each other or with other systems).

Step 1: The first step is to append a singlet held between a and b to the state (12)

$$(\alpha|0\rangle_A + \beta|1\rangle_A)(|00\rangle_{ab} + |11\rangle_{ab})(\gamma|0\rangle_B + \delta|1\rangle_B). \quad (14)$$

Then a local CNOT is performed with A as the control bit and a as the target bit. After this, state (14) is

$$\begin{aligned}&[(\alpha|00\rangle_{Ab} + \beta|11\rangle_{Ab})|0\rangle_a\\ &+ (\alpha|01\rangle_{Ab} + \beta|10\rangle_{Ab})|1\rangle_a](\gamma|0\rangle_B + \delta|1\rangle_B). \quad (15)\end{aligned}$$

Alice then performs a measurement on qubit a, and the line corresponding to this qubit terminates. The measurement result from M reveals one c-bit of information, which is communicated to Bob. If the result is $|0\rangle_a$, Bob does nothing, and if the result is $|1\rangle_a$, Bob performs the X operation on qubit b. After these operations, the state is

$$(\alpha|00\rangle_{Ab} + \beta|11\rangle_{Ab})(\gamma|0\rangle_B + \delta|1\rangle_B). \quad (16)$$

Step 2: Now Bob performs a CNOT on qubits b and B, then a Hadamard on b. Eq.(16) will be

$$\begin{aligned}&\alpha|0\rangle_A(|0\rangle_b + |1\rangle_b)(\gamma|0\rangle_B + \delta|1\rangle_B)\\ &+ \beta|1\rangle_A(|0\rangle_b - |1\rangle_b)(\gamma|1\rangle_B + \delta|0\rangle_B)\\ =&[\alpha|0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B) + \beta|1\rangle_A(\gamma|1\rangle_B + \delta|0\rangle_B)]|0\rangle_b\\ &+ [\alpha(Z|0\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B)\\ &+ \beta(Z|1\rangle_A)(\gamma|1\rangle_B + \delta|0\rangle_B)]|1\rangle_b, \quad (17)\end{aligned}$$

and then Bob measures b in the computation basis, at which point the line denoting b terminates. The result of the measurement (one c-bit) is transmitted to Alice. If the result is $|0\rangle_b$, Alice does nothing, and if the result is $|1\rangle_b$, she performs a Z operation on her qubit A. This completes the non-local CNOT.

This operation has also been found in Refs.[16] and [17]. Reference [16] indicates that two bits of classical communication in each direction and two shared e-bits are necessary and sufficient for the non-local implementation of a quantum CNOT gate. In our scheme, one c-bit in each direction and one shared e-bit are enough for the non-local implementation of a quantum CNOT gate.

4. Conclusion

A distributed quantum computer composed of many local processors requires only a small number of qubits to be stored at each site, which may be experimentally more feasible than storing a large number of qubits in a single site. However, this raises the issue of the non-local implementation of quantum gates. We have elucidated this question and have presented explicitly optimal quantum constructions for the non-local implementation of a CNOT gate and single-qubit rotation around the $\hat{y}$-axis, which use only local operations, prior entanglement, and projective measurements in the computation basis. We have also elucidated some issues concerning the entanglement and classical communication resources required in implementing these actions. A special feature of our scheme is that it consumes less overall resources than the schemes presented in Refs.[14]–[17].

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