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New solitary wave solutions for nonlinear evolution equations*

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Three important nonlinear evolution equations are solved with the aid of the symbolic manipulation system, Maple, using the direct algebraic method proposed recently. We explicitly obtain several new solutions of physical interest in addition to rederiving all the known solutions.

Keywords: nonlinear evolution equation, Riccati equations, solitary wave solution, symbolic computation

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1. Introduction

Recently, the authors of Ref.[1] have presented a direct algebraic method for finding the solitary wave solutions to nonlinear evolution equations. Using this method, one can obtain analytical solutions not only with the form tanh, $\tanh^2$ and sech^2, but also with the form sech and sech×tanh. This approach is extensively used for ordinary as well as partial differential equations, and it can serve as a general recipe for single solitary wave solutions. The key idea of the method is to take full advantage of a coupled Riccati equation, rather than a single Riccati equation, and to use its solutions and the relations between the solutions for solving the desired nonlinear evolution equations, and to obtain their solitary wave solutions and soliton solutions. In this paper, with the aid of the computer algebraic system, Maple, three important nonlinear evolution equations are solved using this method. We obtain some new solutions of physical interest, in addition to rederiving all the known solutions.

2. Method for finding solitary wave solutions

Given an evolution equation, say, in two independent variables

$$Q(\psi, \psi_x, \psi_t, \psi_{xx}, \psi_{tt}, \ldots) = 0, \qquad (1)$$

to begin with, by using the following simple transformation $\xi = x - ct$, where ξ is the travelling wave variable and c is the velocity of the travelling wave, Eq.(1) becomes an ordinary differential equation

$$Q(\psi, \psi', \psi'', \ldots) = 0, \qquad (2)$$

where $' := \mathrm{d}/\mathrm{d}\xi$.

The tanh-function method[2−4] and the extended tanh-function method[5,6] are effective straightforward methods to construct the exact solutions of Eq.(2). However, since the tanh-function method starts from the series expansion in the tanh function, it misses such important types of solitary wave solutions as sech, sech×tanh, etc. We notice that the key idea of the tanh-function method is to use a property of the tanh function, that is, $\tanh(\xi)$ is a solution of the Riccati equation $u' = 1 - u^2$. However, the method used in this paper is to try to seek the solitary wave solutions for nonlinear evolution equations by using the solutions of a coupled Riccati equation rather than a single Riccati equation. The desired coupled Riccati equation reads

$$u' = -kuv, \quad v' = k(1 - v^2), \qquad (3)$$

which admits two types of elementary solitary wave solution

$$u = \pm\mathrm{sech}(k\xi), \quad v = \tanh(k\xi), \qquad (4)$$

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$$u = \pm\mathrm{csch}(k\xi), \quad v = \coth(k\xi). \tag{5}$$

It is found that these two types of solution satisfy the relations $v^2 = 1 - u^2$ and $v^2 = 1 + u^2$, respectively.

The next crucial step is that the solution to Eq.(2) we are looking for is expressed in the form

$$\psi(\xi) = \sum_{i=0}^{m} a_i u^i + \sum_{i=1}^{m} b_i v u^{i-1}, \tag{6}$$

where a_i and b_i are constants to be determined. The parameter m will be found by balancing the linear terms of the highest order with the nonlinear terms in Eq.(2).

Substituting Eq.(6) into Eq.(2), and using Eq.(3) repeatedly, we can express all derivatives of u and v in terms of series in u and v, and we can eliminate any powers of v higher than the first with $v^2 = 1 - u^2$ or $v^2 = 1 + u^2$. Because the coefficients of u^i and vu^i have to vanish, we obtain two sets of algebraic equations (AEs) with respect to a_i, b_i and k, c. Thus, in order to find the solitary waves of the desired Eq.(2) we have to solve the AEs.

The outputs from a computer for solving the algebraic system comprise a list of the form $\{k, c, a_0, a_1, b_1, \ldots, a_m, b_m\}$. In general, if k or any of the parameters is left unspecified, then it is regarded as arbitrary for the solution of Eq.(1).

In the next sections, we solve three important nonlinear evolution equations using the method outlined above.

3. Example A

In this section, we start from the modified Kawachara equation and then, from the results of this, we easily obtain the solutions of modified Benjamin–Bona–Mahony (BBM) equation.

We first consider the following modified Kawachara[7] equation, which describes the motion of water waves with surface tension

$$\psi_t + \psi_x + \psi^2\psi_x + p\psi_{xxx} + q\psi_{xxxxx} = 0, \tag{7}$$

where p and q are all free parameters. To look for the solitary wave solutions of Eq.(7), we make the transformations $\psi(x,t) = \psi(\xi), \xi = x - ct$. Then Eq.(7) turns into the following ordinary differential equation

$$q\psi'''''' + p\psi''' + \psi^2\psi' + \psi' - c\psi' = 0. \tag{8}$$

To begin with, we suppose that the polynomial degree of ψ in (u, v) is m. Balancing $\psi^2\psi'$ with ψ'''''', if $q \neq 0$, or with ψ''', if $q = 0$, we can obtain $m = 2$ or $m = 1$, respectively. This suggests the following ansatz

$$\psi = a_0 + a_1 u + a_2 u^2 + b_1 v + b_2 uv, \tag{9}$$

where one of the coefficients of a_1, a_2, b_1, b_2 is not equal to zero. Applying the rules as before, we can have two sets of algebraic equations corresponding to the relations $v^2 = 1 - u^2$ and $v^2 = 1 + u^2$ respectively.

3.1. Regular solitary wave solutions

To acquire regular solitary wave solutions of Eq.(7), a set of algebraic equations that include thirteen algebraic equations corresponding to the relation $v^2 = 1 - u^2$ is given here:

$$\begin{cases}
-2b_2^3 + 720b_2k^4q + 6a_2^2b_2 = 0,\\
3b_2^3 + 4b_2a_1^2 - 5b_2a_2^2 + 8b_1a_1a_2 - 840k^4qb_2 + 8b_2a_0a_2 - 4b_2b_1^2 - 24k^2pb_2 = 0,\\
-32k^4qa_2 - 8k^2pa_2 + 2ca_2 - 2b_2^2a_0 - 2a_0a_1^2 - 2a_2 - 2a_2b_1^2 - 2a_2a_0^2 - 4a_1b_1b_2 + 2a_0b_1^2 = 0,\\
-k^4qa_1 - k^2pa_1 - a_1 - 2b_2a_0b_1 + ca_1 - a_1b_1^2 - a_1a_0^2 = 0,\\
-2a_2^3 - 720a_2k^4q + 6a_2b_2^2 = 0,\\
4a_2b_1^2 - 4a_2b_2^2 + 24k^2pa_2 - 4a_1^2a_2 + 480k^4qa_2 + 4b_2^2a_0 + 8a_1b_1b_2 - 4a_2^2a_0 = 0,\\
-3a_1b_2^2 - 6a_1a_0a_2 + 3a_1b_1^2 - 6a_2b_2b_1 - a_1^3 + 6k^2pa_1 + 6b_2a_0b_1 + 60k^4qa_1 = 0,\\
10a_2b_2b_1 - 120k^4qa_1 + 5a_1b_2^2 - 5a_2^2a_1 = 0,\\
10b_2a_1a_2 + 5b_1a_2^2 + 120k^4qb_1 - 5b_2^2b_1 = 0,\\
-k^4qb_2 - k^2pb_2 - b_2a_0^2 + cb_2 - b_2 - 2a_1a_0b_1 - b_2b_1^2 = 0,\\
182k^4qb_2 + 20k^2pb_2 - b_2^3 - 3b_2a_1^2 + 2b_2 + 4a_1a_0b_1 - 6b_2a_0a_2 - 6b_1a_1a_2 + 5b_2b_1^2 + 2b_2a_0^2 - 2cb_2 = 0,\\
16k^4qb_1 + 4k^2pb_1 - cb_1 - 4b_2a_0a_1 + b_1a_0^2 - 4b_1a_0a_2 - 2b_2^2b_1 + b_1 - 2b_1a_1^2 + b_1^3 = 0,\\
3b_1a_1^2 + 7b_2^2b_1 - 4b_1a_2^2 + 6b_2a_0a_1 - b_1^3 - 6k^2pb_1 - 120k^4qb_1 - 8b_2a_1a_2 + 6b_1a_0a_2 = 0.
\end{cases} \tag{10}$$

Solving this system of equations (10) by using the Ritt–Wu elimination directly with the aid of Maple, four sets of nontrival solutions to Eqs.(10) are obtained:

$$\left\{a_0 = \frac{p+20qk^2}{\sqrt{-10q}}, a_2 = 6\sqrt{-10q}k^2, \quad c = 1-24qk^4-\frac{p^2}{10q}, a_1 = b_1 = b_2 = 0, k = k\right\}, \tag{11}$$

$$\left\{b_2 = \pm 6p\sqrt{\frac{1}{10q}}, c = 1+\frac{11p^2}{100q}, \quad k = \pm\sqrt{\frac{p}{10q}}, a_0 = a_1 = a_2 = b_1 = 0\right\}, \tag{12}$$

$$\left\{b_1 = \pm\sqrt{-6p}k, c = 1-2pk^2, \quad a_0 = a_1 = a_2 = b_2 = q = 0, k = k\right\}, \tag{13}$$

$$\left\{a_1 = \pm\sqrt{6p}k, c = 1+pk^2, \quad a_0 = a_2 = b_1 = b_2 = q = 0, k = k\right\}. \tag{14}$$

From Eqs.(11), (12), (9) and (4) we obtain two sets of regular solitary wave solutions of Eq.(7) as follows

$$\psi(x,t)_1 = \frac{p+20qk^2}{\sqrt{-10q}} + 6\sqrt{-10q}k^2\mathrm{sech}^2[k(x-ct)], \tag{15}$$

where $c = 1-24qk^4-\dfrac{p^2}{10q}$. In order that the solution is localized, it is necessary that $q<0$.

$$\psi(x,t)_2 = \pm 6p\sqrt{\frac{1}{10q}}\tanh[k(x-ct)]\mathrm{sech}[k(x-ct)], \tag{16}$$

where $c = 1+\dfrac{11p^2}{100q}, k = \pm\sqrt{\dfrac{p}{10q}}$. To guarantee that this solution is localized, the parameters must satisfy the inequalities $p>0$ and $q>0$.

Equation (16) is a new soliton solution to Eq.(17), which, to our knowledge, has not yet been reported in any reference papers. More importantly, it cannot be obtained by using the tanh-function method.[2] Yet another solution (15) is essentially the same as that given in Ref.[8] if we set $\mathrm{sech}^2 x = 1-\tanh^2 x$.

This indicates that the modified Kawachara equation possesses solutions not only with a bell-shaped form, but also with a kink-shaped form. That is to say, solution (15) has a symmetric structure with respect to the centre ($\xi = x-ct = 0$), whereas the structure of solution (16) is antisymmetric. But both of these are important stable solitary wave solutions.

We notice that, in solutions (13) and (14), the parameter $q=0$, as a matter of fact. In this case, the ordinary differential equation (8) becomes

$$p\psi''' + \psi^2\psi' + \psi' - c\psi' = 0. \tag{17}$$

This equation can be derived from the following modified BBM[9] equation

$$\psi_t + \psi_x + \psi^2\psi_x + \tilde{p}\psi_{xxt} = 0. \tag{18}$$

As before, if we set $\xi = x-ct$, Eq.(18) becomes

$$-c\tilde{p}\psi''' + \psi^2\psi' + \psi' - c\psi' = 0. \tag{19}$$

If we set $p = -c\tilde{p}$, Eq.(19) is equivalent to Eq.(17). Thus from solutions (13) and (14) we obtain directly two sets of regular solitary wave solutions to the modified BBM equation.

One solution is

$$\psi(x,t)_1 = \pm\sqrt{\frac{6\tilde{p}}{1-2\tilde{p}k^2}}k\tanh[k(x-ct)], \tag{20}$$

where $c = \dfrac{1}{1-2\tilde{p}k^2}$ with k being an arbitrary constant.

This solution, representing a shock wave structure, is essentially the same as that given in Ref.[8] obtained by using the tanh-function method.

Another new soliton solution can be found as

$$\psi(x,t)_2 = \pm\sqrt{\frac{-6\tilde{p}}{1+\tilde{p}k^2}}k\mathrm{sech}[k(x-ct)], \tag{21}$$

where $c = \dfrac{1}{1+\tilde{p}k^2}$ with k being an arbitrary constant.

3.2. Singular solitary wave solutions

Likewise, we can obtain another set of algebraic equations omitted here, corresponding to the relation $u^2 = 1+v^2$. Solving this, we can obtain singular solitary wave solutions of Eq.(7). For example, when $p<0, q<0$, Eq.(7) admits

$$\psi(x,t)_3 = \pm\frac{6p}{\sqrt{-10q}}\mathrm{csch}\left[\sqrt{\frac{p}{10q}}(x-ct)\right] \times \coth\left[\sqrt{\frac{p}{10q}}(x-ct)\right], \tag{22}$$

where $c = \dfrac{11p^2}{100q}+1$.

Such singular solitary wave solutions are, in fact, diverging shock-typed solutions, which represent a collapsible state and develop singularity at a finite point,

i.e. for any fixed $t = t_0$, there exist x_0 at which these solutions blow up. There is much current interest in the formation of so-called "hot spots" or "blow-up" of solutions.[10,11] It appears that these singular solutions will model these physical phenomena.

4. Example B

In this section, using the same method, we study the combined KdV–mKdV equation[12]

$$\psi_t + (p + q\psi)\psi\psi_x + \psi_{xxx} = 0, \tag{23}$$

where $p, q \neq 0$ are arbitrary parameters.

As for example A, we let $\psi(x,t) = \psi(\xi)$, and Eq.(23) becomes

$$\psi''' - c\psi' + p\psi\psi' + q\psi^2\psi' = 0. \tag{24}$$

Here we suppose Eq.(24) has a solution with the form as Eq.(6). Then by balancing ψ''' with $\psi^2\psi'$, we have $m = 1$. This suggests the following ansatz

$$\psi = a_0 + a_1 u + b_1 v, \tag{25}$$

where $a_1 \neq 0$ or $b_1 \neq 0$.

Substituting Eq.(25) into Eq.(24) we obtain a set of algebraic systems for $a_0, a_1, b_1, q, p, r, k$ and c, corresponding to the relation of $v^2 = 1 - u^2$, namely

$$\begin{cases} -b_1 a_1 (p + 2a_0 q) = 0, \\ b_1 (q b_1^2 + 4rk^2 + q a_0^2 + p a_0 - c - 2q a_1^2) = 0, \\ b_1 (-q b_1^2 - 6rk^2 + 3q a_1^2) = 0, \\ 2b_1 a_1 (p + 2a_0 q) = 0, \\ -a_1 (rk^2 + q b_1^2 - c + p a_0 + q a_0^2) = 0, \\ -(a_1 - b_1)(a_1 + b_1)(p + 2a_0 q) = 0, \\ -a_1 (q a_1^2 - 6rk^2 - 3q b_1^2) = 0. \end{cases} \tag{26}$$

Solving Eq.(26) we obtain two sets of general solutions

$$\left\{ a_0 = -\frac{p}{2q}, a_1 = 0, b_1 = \pm\frac{k\sqrt{-6q}}{q}, \right. \\ \left. c = -2k^2 + \frac{p^2}{4q}, k = k \right\}, \tag{27}$$

and

$$\left\{ a_0 = -\frac{p}{2q}, a_1 = \pm\frac{k\sqrt{6q}}{q}, \right. \\ \left. c = k^2 - \frac{p^2}{4q}, b_1 = 0, k = k \right\}. \tag{28}$$

From Eqs.(27), (28), (25) and (4) we can obtain two sets of regular solitary wave solutions of Eq.(23) as follows.

For $q < 0$, we have

$$\psi(x,t)_1 = -\frac{p}{2q} \pm \frac{k\sqrt{-6qr}}{q} \tanh[k\,(x - ct)], \tag{29}$$

where $c = -2k^2 + \dfrac{p^2}{4q}$ with k being an arbitrary constant. This solution is substantially the same as that given in Ref.[8] obtained by using the tanh-function method.

For $q > 0$, we have

$$\psi(x,t)_2 = -\frac{p}{2q} \pm \frac{k\sqrt{6qr}}{q} \mathrm{sech}[k(x - ct)], \tag{30}$$

where $c = k^2 - \dfrac{p^2}{4q}$ with k being an arbitrary constant. This is a new solution to Eq.(23) obtained by using the present method, which cannot be obtained using the tanh-function method.

Similarly, we can also gain two sets of singular solitary wave solutions of Eq.(23) as follows.

For $q < 0$, we have

$$\psi(x,t)_3 = -\frac{p}{2q} \pm \frac{k\sqrt{-6qr}}{q} \coth[k\,(x - ct)], \tag{31}$$

where $c = -2k^2 + \dfrac{p^2}{4q}$ with k being an arbitrary constant.

For $q > 0$, we have

$$\psi(x,t)_4 = -\frac{p}{2q} \pm \frac{k\sqrt{6qr}}{q} \mathrm{csch}[k(x - ct)], \tag{32}$$

where $c = k^2 - \dfrac{p^2}{4q}$ with k being an arbitrary constant.

5. Summary

To summarize, we have reported on some regular solitary wave solutions and singular solitary wave solutions for three important nonlinear evolution equations by using the coupled Riccati equation and symbolic computation. For each of these, we have respectively obtained an important new solution and we verified these by putting them back into the original equations using Maple.

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