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Existential theorem of conserved quantities and its inverse for the dynamics of nonholonomic relativistic systems

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We present a general approach to the construction of conservation laws for the dynamics of nonholonomic relativistic systems. Firstly, we give the definition of integrating factors for the differential equations of motion of a mechanical system. Next, the necessary conditions for the existence of the conserved quantities are studied in detail. Then, we establish the existential theorem for the conserved quantities and its inverse for the equations of motion of a nonholonomic relativistic system. Finally, an example is given to illustrate the application of the result.

Keywords: nonholonomic system, relativity, integrating factor, conserved theorem
PACC: 0320, 0200

1. Introduction

The conservation laws for a dynamical system play an important role in the study of mechanics. In particular, the existence of a certain conserved quantity will help us to understand the partial physical state and quality of the system studied when it is hard to solve the differential equations of motion of the system. Furthermore, this has become one of the most important directions in modern analytical mechanics. There are two main methods used to seek the conserved quantity. One method is based on the theory of the invariance of Hamilton's action under infinitesimal transformation, namely, the Noether theory.[1] The other method was given by Lutzky[2] in 1979 who introduced Lie's method of the expansive group of invariance for ordinary differential equations under infinitesimal transformations to the mechanical system and presented the Lie symmetries of differential equations of motion. Significant achievements have been obtained in the study of this method in recent years.[3−12]

Relativistic analytical mechanics are the foundation of modern physics. In this paper we seek the conserved quantity of a nonholonomic relativistic system on the basis of the theory of integrating factors.[13] Compared with the two famous methods mentioned above, this method has the benefits of fewer conditions, easy calculation and wide practical applications. It will also have a prosperous prospect in the integration theory of relativistic analytical mechanics. Firstly, we give the definition of integrating factors for differential equations of motion and we study the necessary conditions for the existence of conserved quantities. Secondly, we establish the conservation theorem and its inverse theorem for the equations of motion of a nonholonomic relativistic system. Finally, we present an example to illustrate the application of the results.

2. Hamilton's canonical equations of motion of relativistic systems

We consider a mechanical system composed of N particles, and its configuration is determined by n generalized coordinates $q_1, q_2, \cdots, q_n$. The motion of the system is subjected to the following g ideal bilateral nonholonomic constraints of Chetaev's type

$$f_\beta(q, \dot{q}, t) = 0, \quad \frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0$$

$$(\beta = 1, 2, \cdots, g; s = 1, 2, \cdots, n). \tag{1}$$

The position vector of a representative point i of the system is $\boldsymbol{r}_i(q_1, q_2, \cdots, q_n, t)$, and its mass can be written as

$$m_i = \frac{m_{0i}}{\sqrt{1 - \dot{\boldsymbol{r}}_i^2/c^2}}. \tag{2}$$

The generalized kinetic energy function[7] of a relativistic system is

$$T_r^* = m_{0i}c^2(1-\sqrt{1-\dot{\boldsymbol{r}}_i^2/c^2})$$
$$(i=1,2,\cdots,N). \tag{3}$$

The equation of motion of the system can be written in Routh's form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial \dot{q}_s}-\frac{\partial T_r^*}{\partial q_s}=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}$$
$$(s=1,2,\cdots,n). \tag{4}$$

Dividing all the active forces acting on the system into nonpotential forces and potential forces, we have

$$Q_s=-\frac{\partial V}{\partial q_s}+Q'_s(q_k,\dot{q}_k,t), V=V(q_k)$$
$$(s,k=1,2,\cdots,n). \tag{5}$$

We construct Lagrange's function of the relativistic system as

$$L_r^*=T_r^*-V, \tag{6}$$

then Eqs.(4) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r^*}{\partial \dot{q}_s}-\frac{\partial L_r^*}{\partial q_s}=Q'_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}$$
$$(s=1,2,\cdots,n), \tag{7}$$

where λ_β are the constrained multipliers. Before we integrate the differential equations of motion, we obtain λ_β from Eqs.(1) and (4), and express it as a function of t, q and $\dot{q}$. Thus, Eqs.(7) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r^*}{\partial \dot{q}_s}-\frac{\partial L_r^*}{\partial q_s}=Q'_s+\Lambda_s$$
$$(s=1,2,\cdots,n), \tag{8}$$

where $\Lambda_s=\lambda_\beta(t,q,\dot{q})\dfrac{\partial f_\beta}{\partial \dot{q}_s}$ are the generalized nonholonomic reactive forces of the constraint, and have been written as functions of the variables t, q and $\dot{q}$.

Equations (8) are called differential equations of motion of the holonomic systems corresponding to the nonholonomic systems (1) and (7). The integrals of the nonholonomic systems (1) and (7) can be found according to the corresponding holonomic system (8), only if we apply the restriction of the nonholonomic constraints (1) to the initial conditions.

Now, we introduce the generalized momentum and Hamilton's function as

$$p_s=\frac{\partial L_r^*}{\partial \dot{q}_s},$$
$$H_r(t,q,p)=p_s\dot{q}_s-L_r^* \quad (s=1,2,\cdots,n). \tag{9}$$

From Eqs.(8) and (9), we have

$$\dot{q}_s=\frac{\partial H_r}{\partial p_s}, \tag{10}$$

$$\dot{p}_s=-\frac{\partial H_r}{\partial q_s}+\widetilde{Q}'_s+\widetilde{\Lambda}_s \quad (s=1,2,\cdots,n). \tag{11}$$

Here, the symbol "$\sim$" is the expression, which is obtained by substituting t, q, p for $\dot{q}$.

Equations (10) and (11) are called Hamilton's canonical equations of motion of holonomic systems corresponding to the nonholonomic relativistic systems (1) and (7).

3. Integrating factors and existential theorems of the conserved quantities

3.1. Integrating factors

If the invariant $\left(\dot{p}_s+\dfrac{\partial H_r}{\partial q_s}-\widetilde{Q}'_s-\widetilde{\Lambda}_s\right)G_s$ reduces identically to

$$\begin{aligned}&\left(\dot{p}_s+\frac{\partial H_r}{\partial q_s}-\widetilde{Q}'_s-\widetilde{\Lambda}_s\right)G_s\\ \equiv&\frac{\mathrm{d}}{\mathrm{d}t}(p_sG_s-H_rR-B)\\ &+\mu_s\left(\dot{p}_s+\frac{\partial H_r}{\partial q_s}-\widetilde{Q}'_s-\widetilde{\Lambda}_s\right),\end{aligned} \tag{12}$$

where R, B and μ_s are functions of t, q and p. In this case, functions $G_s=G_s(t,q,p)$ are called integrating factors of the canonical equations of motion (11).

Combining Eqs.(11) with (12), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(p_sG_s-H_rR-B)\\ =&-\mu_s\left(\dot{p}_s+\frac{\partial H_r}{\partial q_s}-\widetilde{Q}'_s-\widetilde{\Lambda}_s\right),\end{aligned} \tag{13}$$

and the following theorem can be obtained.

3.2. Existential theorems of the conserved quantities

Theorem 1. If functions G_s are integrating factors of Eqs.(11), then there exists a conserved quantity (first integration)

$$D=p_sG_s-H_rR-B \tag{14}$$

of the holonomic systems (10) and (11), corresponding to the nonholonomic relativistic systems (1) and (7).

For a given dynamical system, if functions G_s are integrating factors of Eqs.(11), the necessary condition (13) must be satisfied for every set of functions G_s, R, B and μ_s. The condition can be written as

$$-R\frac{\partial H_r}{\partial t}+(\widetilde{Q}'_s+\widetilde{\Lambda}_s)\left(G_s-R\frac{\partial H_r}{\partial p_s}\right)+p_s\dot{G}_s-\frac{\partial H_r}{\partial q_s}G_s-H_r\dot{R}-\dot{B}+I=0, \quad (15)$$

where we have used Eqs.(10) and (11) to calculate $\dot{H}_r$, and

$$I=\mu_s\left(\dot{p}_s+\frac{\partial H_r}{\partial q_s}-\widetilde{Q}'_s-\widetilde{\Lambda}_s\right). \quad (16)$$

If a set of functions G_s, R, B and μ_s satisfies the necessary condition (15), and the right-hand side of Eq.(14) is reduced to a constant, then we have a singular set of functions G_s, R, B and μ_s.

From these considerations and from Theorem l, we can formulate the following theorem, convenient for finding the first integrals of the canonical equations of motion (10) and (11).

Theorem 2. For every nonsingular set of functions G_s, R, B and μ_s, which satisfies the necessary condition (15), there exists a conserved quantity (14) of the holonomic systems (10) and (11), corresponding to the nonholonomic relativistic systems (1) and (7).

By integration of Eq.(15), the nonsingular set of functions G_s, R, B and μ_s, which gives a conserved quantity, can be obtained. In any case, if we have any solution of Eq.(15), which does not contain any integration constant, we call the solution a functional solution of Eq.(15). When a nonsingular functional solution of Eq.(15) is substituted into Eq.(14), we obtain an ordinary first integral of the dynamical system. Here, the only integration constant is the value of D. Vujanovic[14] has shown that by using a complete solution of Eq.(15), which contains a sufficient number of integration constants, we can finally form a finite solution of the equations of motion (10) and (11).

Obviously, in Eq.(15) $\mathrm{d}(\ )/\mathrm{d}t$ must be interpreted as the following operator

$$\frac{\mathrm{d}(\ )}{\mathrm{d}t}=\frac{\partial(\ )}{\partial t}+\frac{\partial H_r}{\partial p_s}\frac{\partial(\ )}{\partial q_s}+\dot{p}_s\frac{\partial(\ )}{\partial p_s}. \quad (17)$$

4. Killing equations

It is critical to find functions $R=R(t,q,p)$, $B=B(t,q,p)$ and $G_s=G_s(t,q,p)$ when using the theory of integrating factors to seek the conserved quantity of a dynamical system. Usually, we expand the necessary condition (15) and split it into first-order partial differential equations for G_s, R and B. This type of partial differential equation is called the Killing equation. We can obtain the above functions, through solving Killing equations.

4.1. We can expand Eq.(15) by using Eqs.(16) and (17). We divide it into terms with and without $\dot{p}_\beta$, and set them equal to zero separately. Then we have

$$-R\frac{\partial H_r}{\partial t}+(\widetilde{Q}'_s+\widetilde{\Lambda}_s)\left(G_s-R\frac{\partial H_r}{\partial p_s}\right)+p_s\left(\frac{\partial G_s}{\partial t}+\frac{\partial G_s}{\partial q_\beta}\frac{\partial H_r}{\partial p_\beta}\right)-\frac{\partial H_r}{\partial q_s}G_s-H_r\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial q_\beta}\frac{\partial H_r}{\partial p_\beta}\right)-\frac{\partial B}{\partial t}-\frac{\partial B}{\partial q_\beta}\frac{\partial H_r}{\partial p_\beta}+\mu_\beta\left(\frac{\partial H_r}{\partial q_\beta}-\widetilde{Q}'_\beta-\widetilde{\Lambda}_\beta\right)=0, \quad (18)$$

$$p_s\frac{\partial G_s}{\partial p_\beta}-H_r\frac{\partial R}{\partial p_\beta}-\frac{\partial B}{\partial p_\beta}+\mu_\beta=0. \quad (19)$$

If the system admits a nonsingular solution with respect to $R=R(t,q,p)$, $B=B(t,q,p)$, $G_s=G_s(t,q,p)$ and $\mu_s=\mu_s(t,q,p)$, then a first integral (14) exists automatically. Here, we have a system of $(n+1)$ linear partial differential equations with, in the most general cases, $(2n+2)$ unknown functions. Thus, there is a great freedom of their solution. The most logical suggestion is to prescribe $(n+1)$ functions in advance and, after that, to solve the equations with respect to the remaining $(n+1)$ unknowns. Usually, the prescribed functions are B and μ_s.

4.2. We can expand the necessary condition (15) using Eqs.(11), (16) and (17), to obtain

$$-R\frac{\partial H_r}{\partial t}+(\widetilde{Q}'_s+\widetilde{\Lambda}_s)\left(G_s-R\frac{\partial H_r}{\partial p_s}\right)+p_s\left(\frac{\partial G_s}{\partial t}+\frac{\partial G_s}{\partial q_\beta}\frac{\partial H_r}{\partial p_\beta}\right)-\frac{\partial H_r}{\partial q_s}G_s-H_r\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial q_\beta}\frac{\partial H_r}{\partial p_\beta}\right)-\frac{\partial B}{\partial t}-\frac{\partial B}{\partial q_\beta}\frac{\partial H_r}{\partial p_\beta}+\left(\widetilde{Q}'_\beta+\widetilde{\Lambda}_\beta-\frac{\partial H_r}{\partial q_\beta}\right)\times\left(p_s\frac{\partial G_s}{\partial p_\beta}-H_r\frac{\partial R}{\partial p_\beta}-\frac{\partial B}{\partial p_\beta}\right)=0. \quad (20)$$

Equation (20) is a linear partial differential equation. In the general case, each of the $(n+2)$ functions G_s, R and B can be regarded as unknown. According to Theorem 2, any nonsingular solution of Eq.(20) with respect to G_s, R and B yields a first integral (14).

5. Inverse theorems of theorems 1 and 2

The connection between the first integral (14) of the dynamical system, based on the theory of integration, and the corresponding integrating factors G_s and functions R and B must be consistent with the necessary condition (15). It is shown that the necessary condition (15) is equivalent to Eqs.(18), (19) or (20). Therefore, the set of $(n+2)$ functions G_s, R and B must satisfy Eqs.(14), (18) and (19) or (14) and (20).

Calculating $\partial B/\partial p_\beta$ from Eq.(14), and substituting this result into Eq.(19), we have

$$G_s = \frac{\partial D}{\partial p_s} + R\frac{\partial H_r}{\partial p_s}. \tag{21}$$

Now, combining Eqs.(14) and (21), we obtain

$$B = p_s\frac{\partial D}{\partial p_s} - D + R\left(p_s\frac{\partial H_r}{\partial p_s} - H_r\right). \tag{22}$$

Next, letting the first integral $D = D(t,q,p)$ equal the conserved quantity, we have

$$p_s G_s - H_r R - B = D(t,q,p). \tag{23}$$

From Eqs.(21), (22) and (23), G_s, R and B can be obtained.

In this way, we formulate the following Theorem.

Theorem 3. If every constant of the motion $D(t,q,p)$ for a corresponding holonomic system can be described by a Hamilton's function H_r, generalized nonpotential forces $\widetilde{Q}'_s$ and generalized constraint reaction forces $\widetilde{\Lambda}_s$, the integrating factors G_s and functions R and B corresponding to D are determined by Eqs.(21), (22) and (23).

6. Example

Suppose Lagrange's function of a relativistic system is

$$L_r = m_0c^2\left(1 - \sqrt{1 - \frac{\dot{q}_1^2 + \dot{q}_2^2}{c^2}}\right) - V. \tag{24}$$

The generalized nonpotential forces acting on the system are $Q_1 = \dot{q}_1, Q_2 = -\dot{q}_2$. The potential energy function V of the system is constant. The motion of the system is subjected to the following ideal bilateral nonholonomic constraint of Chetaev's type

$$f = \dot{q}_1 - \dot{q}_2 + q_1 t = 0. \tag{25}$$

We try to find the conserved quantities of the system.

Firstly, we study the direct problem to obtain the conserved quantities of the system.

The first step is to establish Hamilton's canonical equations (10) and (11) corresponding to the holonomic system. Because we have

$$p_1 = \frac{\partial L_r}{\partial \dot{q}_1} = \frac{m_0\dot{q}_1}{\sqrt{1 - \frac{\dot{q}_1^2 + \dot{q}_2^2}{c^2}}} = m\dot{q}_1,$$

$$p_2 = \frac{\partial L_r}{\partial \dot{q}_2} = \frac{m_0\dot{q}_2}{\sqrt{1 - \frac{\dot{q}_1^2 + \dot{q}_2^2}{c^2}}} = m\dot{q}_2, \tag{26}$$

where

$$m = \frac{m_0}{\sqrt{1 - \frac{\dot{q}_1^2 + \dot{q}_2^2}{c^2}}}.$$

From Eqs.(8) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_1) = \dot{q}_1 + \lambda,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_2) = -\dot{q}_2 - \lambda. \tag{27}$$

From Eqs.(25) and (27), we obtain

$$\lambda = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \dot{q}_2 + \frac{q_1t}{2}. \tag{28}$$

Therefore, we obtain

$$Q'_1 + \Lambda_1 = \dot{q}_1 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \dot{q}_2 + \frac{q_1t}{2},$$

$$Q'_2 + \Lambda_2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \frac{q_1t}{2}, \tag{29}$$

$$\dot{q}_1 = \frac{p_1}{m}, \quad \dot{q}_2 = \frac{p_2}{m}. \tag{30}$$

The Hamilton's function is

$$H_r = \frac{p_1^2 + p_2^2}{m} - m_0c^2\left(1 - \sqrt{1 - \frac{\dot{q}_1^2 + \dot{q}_2^2}{c^2}}\right) + V. \tag{31}$$

Then, the canonical equations are

$$\dot{q}_1 = \frac{\partial H_r}{\partial p_1} = \frac{p_1}{m}, \quad \dot{q}_2 = \frac{\partial H_r}{\partial p_2} = \frac{p_2}{m}. \tag{32}$$

$$\begin{aligned}\dot{p}_1 =& -\frac{\partial H_r}{\partial q_1} + \widetilde{Q}'_1 + \widetilde{\Lambda}_1\\ =& \frac{p_1 - p_2}{m} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) + \frac{q_1t}{2},\end{aligned}$$

$$\begin{aligned}\dot{p}_2 =& -\frac{\partial H_r}{\partial q_2} + \widetilde{Q}'_2 + \widetilde{\Lambda}_2\\ =& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t) - \frac{q_1t}{2}.\end{aligned} \tag{33}$$

The second step is to establish the Killing equation.

Since the Hamilton's function does not contain time t and generalized coordinates q_1 and q_2, from Eq.(20), we have

$$\begin{aligned}
&\left[\frac{p_1-p_2}{m}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t)+\frac{q_1t}{2}\right]\left(G_1-R\frac{\partial H_r}{\partial p_1}\right)\\
&+\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(mq_1t)-\frac{q_1t}{2}\right]\left(G_2-R\frac{\partial H_r}{\partial p_2}\right)\\
&+p_1\left(\frac{\partial G_1}{\partial t}+\frac{\partial G_1}{\partial q_1}\frac{\partial H_r}{\partial p_1}+\frac{\partial G_1}{\partial q_2}\frac{\partial H_r}{\partial p_2}\right)\\
&+p_2\left(\frac{\partial G_2}{\partial t}+\frac{\partial G_2}{\partial q_1}\frac{\partial H_r}{\partial p_1}+\frac{\partial G_2}{\partial q_2}\frac{\partial H_r}{\partial p_2}\right)\\
&-H_r\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial q_1}\frac{\partial H_r}{\partial p_1}+\frac{\partial R}{\partial q_2}\frac{\partial H_r}{\partial p_2}\right)\\
&-\frac{\partial B}{\partial t}-\frac{\partial B}{\partial q_1}\frac{\partial H_r}{\partial p_1}-\frac{\partial B}{\partial q_2}\frac{\partial H_r}{\partial p_2}\\
&+\left[\frac{p_1-p_2}{m}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(-mq_1t)+\frac{q_1t}{2}\right]\\
&\times\left(p_1\frac{\partial G_1}{\partial p_1}+p_2\frac{\partial G_2}{\partial p_1}-H_r\frac{\partial R}{\partial p_1}-\frac{\partial B}{\partial p_1}\right)\\
&+\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(mq_1t)-\frac{q_1t}{2}\right]\\
&\times\left(p_1\frac{\partial G_1}{\partial p_2}+p_2\frac{\partial G_2}{\partial p_2}-H_r\frac{\partial R}{\partial p_2}-\frac{\partial B}{\partial p_2}\right)=0. \quad (34)
\end{aligned}$$

The third step is to seek the conserved quantity of the system.

There are solutions of Eq.(34):

$$R=0,\quad G_1=1,\quad G_2=1,\quad B=q_1-q_2. \quad (35)$$

For the integrating factors G_1 and G_2 and the functions R and B, all satisfying the necessary condition (15), according to Theorem 1, the conserved quantity (14) can be given as

$$\begin{aligned}
D=&m\dot{q}_1+m\dot{q}_2-q_2+q_1\\
=&\frac{m_0(\dot{q}_1+\dot{q}_2)}{\sqrt{1-\dfrac{\dot{q}_1^2+\dot{q}_2^2}{c^2}}}+(q_2-q_1)=\text{const.} \quad (36)
\end{aligned}$$

Secondly, we study the inverse problem, in which integrating factors G_s and functions R and B can be obtained from the known first integral.

Suppose the studied system has a first integral

$$\begin{aligned}
D=&m\dot{q}_1+m\dot{q}_2+(q_2-q_1)\\
=&p_1+p_2+(q_2-q_1)=\text{const.} \quad (37)
\end{aligned}$$

From Eqs.(21) and (22), we have

$$G_1=1+R\frac{p_1}{m},\quad G_2=1+R\frac{p_2}{m},$$

$$R=\frac{B+(q_2-q_1)}{m_0c^2\left(1-\sqrt{1-\dfrac{p_1^2+p_2^2}{m^2c^2}}\right)-V},$$

and when $B=q_1-q_2$, we have

$$R=0,\quad G_1=1,\quad G_2=1.$$

These are consistent with Eq.(35).

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