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Lie symmetries and conserved quantities of Birkhoff systems with unilateral constraints*

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In this paper we study the Lie symmetries of Birkhoff systems with unilateral constraints. We give the conditions for, and the form of, conserved quantities due to the Lie symmetries of the systems, and we also study the inverse problem of the Lie symmetries of the systems. Finally, an example is given to illustrate the application of the results.

Keywords: analytical mechanics, Birkhoff systems with unilateral constraints, Lie symmetry, conserved quantity, inverse problem

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1. Introduction

The conservation laws (or first integrals) of mechanical systems are always of mathematical importance and, at the same time, they are regarded as a manifestation of some profound physical principle. The modern methods in finding conservation laws are mainly the Noether[1] method and the Lie method. In 1918, using the invariance of Hamilton action under infinitesimal transformations, Noether studied conservation laws of mechanical systems and presented the famous Noether's theorem. In 1979, Lutzky[2] introduced again the method of the extended group which Lie used to study the invariance of differential equations at the end of the 19th century. Lutzky investigated the method and posed the Lie symmetries which make the differential equations of motion invariant. In the past twenty years or so, great progress[3−29] has been made in seeking modern methods for finding conservation laws, i.e. the Noether symmetry and the Lie symmetry.

In 2000, Zhang and Mei[21] studied Lie symmetries and conserved quantities of nonholonomic mechanical systems with unilateral constraints. As a Birkhoff system is a generalization of a Hamilton system, the study of the Birkhoff system is a significant aspect of modern analytical mechanics. In this paper, we further study the Lie symmetries of Birkhoff systems with unilateral constraints, we establish the determining equations of the Lie symmetries, and we give the structure equations and the conserved quantities. We study two problems of the systems: that of finding the conserved quantity from a Lie symmetry and that of finding the Lie symmetry from a known integral.

2. Lie symmetries and conserved quantities of Birkhoff systems with unilateral constraints

To study a Birkhoff system, we suppose that the motion of the system is subjected to the unilateral constraints

$$f_\beta(t, a^\mu) \geq 0, \quad (\beta = 1, 2, \cdots, g; \mu = 1, 2, \cdots, 2n). \tag{1}$$

Then the differential equations of motion of the system can be expressed in the form[30]

$$\sum_{\nu=1}^{2n} \Omega_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial f_\beta}{\partial a^\mu},$$
$$(\mu = 1, 2, \cdots, 2n),$$
$$\lambda_\beta \geq 0, \quad f_\beta \geq 0, \quad \lambda_\beta f_\beta = 0,$$
$$(\beta = 1, 2, \cdots, g), \tag{2}$$

where $\Omega_{\mu\nu}$ is the Birkhoffian covariant tensor, B is the Birkhoffian function, R_μ is the Birkhoff's function and λ_β is the constraint multiplier. If the system is in the constrained state, $f_\beta(t, a^\mu) = 0$. In contrast, if the system is divorced from the constraints, i.e. $f_\beta(t, a^\mu) > 0$, then the right-hand side of the first set of Eqs.(2) would be zero.

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We introduce the infinitesimal transformations

$$\begin{cases} t^* = t + \Delta t, \\ a^{\mu *} = a^\mu + \Delta a^\mu, \end{cases} \quad (\mu = 1, 2, \cdots, 2n), \tag{3}$$

or its expanded formulae

$$\begin{cases} t^* = t + \varepsilon \xi_0(t, \boldsymbol{a}), \\ a^{\mu *} = a^\mu + \varepsilon \xi_\mu(t, \boldsymbol{a}), \end{cases} \quad (\mu = 1, 2, \cdots, 2n), \tag{4}$$

where ε is an infinitesimal parameter, and ξ_0 and ξ_μ are generators of the infinitesimal transformations. We also introduce the infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \sum_{\mu=1}^{2n} \xi_\mu \frac{\partial}{\partial a^\mu}. \tag{5}$$

Its first extended infinitesimal generator is

$$X^{(1)} = X^{(0)} + \sum_{\mu=1}^{2n} (\dot{\xi}_\mu - \dot{a}^\mu \dot{\xi}_0) \frac{\partial}{\partial \dot{a}^\mu}. \tag{6}$$

Then, we study the invariance of Eqs.(2) under the infinitesimal transformations (4), that is, Lie symmetries of Birkhoff systems with unilateral constraints.

Firstly, if the system is in the constrained state, that is, constraints (1) take the sign of equality, we have

$$f_\beta(t, a^\mu) = 0, \quad (\beta = 1, 2, \cdots, g; \mu = 1, 2, \cdots, 2n). \tag{7}$$

λ_β can be expressed as an explicit function of t before integrating the differential equations of motion, thus Eqs.(2) can be written as[31]

$$\sum_{\nu=1}^{2n} \Omega_{\mu\nu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = P_\mu, \quad (\mu = 1, 2, \cdots, 2n), \tag{8}$$

where

$$P_\mu = P_\mu(t, \boldsymbol{a}) = \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial f_\beta}{\partial a^\mu}. \tag{9}$$

Expanding Eqs.(8), we have

$$\dot{a}^\mu = \sum_{\nu=1}^{2n} \Omega_{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right), \quad (\mu = 1, 2, \cdots, 2n), \tag{10}$$

where $\Omega^{\mu\nu}$ is the Birkhoffian contravariant tensor. The necessary and sufficient conditions under which Eqs.(10) remain invariant are

$$\dot{\xi}_\mu - \sum_{\nu=1}^{2n} \Omega_{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right) \dot{\xi}_0 = X^{(0)} \left[\sum_{\nu=1}^{2n} \Omega_{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right) \right]. \tag{11}$$

Since the system is in the constrained state, the restriction conditions of the constraints under the vector field (6) are

$$X^{(0)}[f_\beta(t, \boldsymbol{a})] = 0, \quad (\beta = 1, 2, \cdots, g) \tag{12}$$

Secondly, if the system is divorced from the constraints, that is, the sign of the inequality in constraints (1) holds strictly, we have

$$f_\beta(t, a^\mu) > 0, \quad (\beta = 1, 2, \cdots, g; \mu = 1, 2, \cdots, 2n). \tag{13}$$

Then Eqs.(2) become

$$\sum_{\nu=1}^{2n} \Omega_{\mu\nu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} = 0, \quad (\mu = 1, 2, \cdots, 2n). \tag{14}$$

Expanding Eqs.(14), we obtain

$$\dot{a}^\mu = \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right), \quad (\mu = 1, 2, \cdots, 2n). \tag{15}$$

The necessary and sufficient conditions under which Eqs.(15) remain invariant under the infinitesimal transformations (4) are

$$\dot{\xi}_\mu - \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \dot{\xi}_0 = X^{(0)} \left[\sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \right]. \tag{16}$$

The result discussed above can be summed up in the following proposition 1.

Proposition 1: For Birkhoff systems with unilateral constraints (1) and (2), if the infinitesimal transformations (4) satisfy the determining equations

$$\dot{\xi}_\mu - \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right) \dot{\xi}_0 = X^{(0)} \left[\sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right) \right], \quad \text{for } f_\beta = 0, \tag{11}'$$

$$\dot{\xi}_\mu - \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \dot{\xi}_0 = X^{(0)} \left[\sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} \right) \right], \quad \text{for } f_\beta > 0, \tag{16}'$$

and the restriction conditions

$$X^{(0)}[f_\beta(t,\boldsymbol{a})]=0,\quad \text{for } f_\beta=0, \tag{12}'$$

then they are Lie symmetrical transformations.

Hence, we have

Proposition 2: For the Lie symmetrical transformations (4) of Birkhoff systems with unilateral constraints, if there exists a gauge function $G=G(t,\boldsymbol{a})$ satisfying the structure equations

$$\begin{aligned}&\left(\sum_{\mu=1}^{2n}R_\mu\dot{a}^\mu-B\right)\dot{\xi}_0+X^{(1)}\left(\sum_{\mu=1}^{2n}R_\mu\dot{a}^\mu-B\right)\\&-\sum_{\mu=1}^{2n}P_\mu(\xi_\mu-\dot{a}^\mu\xi_0)+\dot{G}=0,\end{aligned}$$

$$\text{for } f_\beta=0, \tag{17}$$

$$\left(\sum_{\mu=1}^{2n}R_\mu\dot{a}^\mu-B\right)\dot{\xi}_0+X^{(1)}\left(\sum_{\mu=1}^{2n}R_\mu\dot{a}^\mu-B\right)+\dot{G}=0,$$

$$\text{for } f_\beta>0, \tag{18}$$

then the system has the conserved quantity

$$I=\sum_{\mu=1}^{2n}R_\mu\xi_\mu-B\xi_0+G=\text{const.} \tag{19}$$

Proof. We have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\sum_{\mu=1}^{2n}\left(\frac{\partial R_\mu}{\partial t}+\sum_{\nu=1}^{2n}\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\right)\xi_\mu+\sum_{\mu=1}^{2n}R_\mu\dot{\xi}_\mu\\&-\left(\frac{\partial B}{\partial t}+\sum_{\mu=1}^{2n}\frac{\partial B}{\partial a^\mu}\dot{a}^\mu\right)\xi_0-B\dot{\xi}_0+\dot{G}.\end{aligned}$$

If the system is in the constrained state, then by utilizing Eqs.(8) and the structure equations (17), we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\sum_{\mu=1}^{2n}\left(\frac{\partial R_\mu}{\partial t}+\sum_{\nu=1}^{2n}\frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\right)\xi_\mu+\sum_{\mu=1}^{2n}R_\mu\dot{\xi}_\mu-\left(\frac{\partial B}{\partial t}+\sum_{\mu=1}^{2n}\frac{\partial B}{\partial a^\mu}\dot{a}^\mu\right)\xi_0-B\dot{\xi}_0-\left(\sum_{\mu=1}^{2n}R_\mu\dot{a}^\mu-B\right)\dot{\xi}_0\\&-\sum_{\mu=1}^{2n}(\dot{\xi}_\mu-\dot{a}^\mu\dot{\xi}_0)R_\mu-\sum_{\mu=1}^{2n}\left(\frac{\partial R_\mu}{\partial t}\xi_0+\sum_{\nu=1}^{2n}\frac{\partial R_\mu}{\partial a^\nu}\xi_\nu\right)\dot{a}^\mu+\frac{\partial B}{\partial t}\xi_0+\sum_{\mu=1}^{2n}\frac{\partial B}{\partial a^\mu}\xi_\mu+\sum_{\mu=1}^{2n}P_\mu(\xi_\mu-\dot{a}^\mu\xi_0)\\=&\sum_{\mu=1}^{2n}(\xi_\mu-\dot{a}^\mu\xi_0)\left[\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}+P_\mu-\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right]-\sum_{\mu=1}^{2n}\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\mu\dot{a}^\nu\xi_0=0.\end{aligned}$$

Also, if the system is divorced from the constraint, then by utilizing Eqs.(14) and the structure equations (18), we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}=&\sum_{\mu=1}^{2n}(\xi_\mu-\dot{a}^\mu\xi_0)\left[\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}+P_\mu\right.\\&\left.-\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right]\\&-\sum_{\mu=1}^{2n}\sum_{\nu=1}^{2n}\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\mu\dot{a}^\nu\xi_0=0.\end{aligned}$$

Therefore, the system has the conserved quantity (19).

3. Inverse problem of Lie symmetries of Birkhoff systems with unilateral constraints

For Birkhoff systems with unilateral constraints, the inverse problem of Lie symmetries is to find the corresponding Lie symmetries according to the known conserved quantity. The solution of the inverse problem of Lie symmetries of the system is as follows. Firstly, according to Noether theory, we can find the corresponding Noether symmetries by the known integral. Secondly, we must verify whether the Noether symmetries obtained above are Lie symmetries. For the Noether symmetries of Birkhoff systems with unilateral constraints (1) and (2), we have [32]

$$\sum_{\mu=1}^{2n}\frac{\partial f_\beta}{\partial a^\mu}(\xi_\mu-\dot{a}^\mu\xi_0)=0, \tag{20}$$

$$\begin{aligned}&\left(\sum_{\mu=1}^{2n}\frac{\partial R_\mu}{\partial t}\dot{a}^\mu-\frac{\partial B}{\partial t}\right)\xi_0+\sum_{\mu=1}^{2n}\left(\sum_{\nu=1}^{2n}\frac{\partial R_\nu}{\partial a^\mu}\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}\xi_\mu\right)\\&-B\dot{\xi}_0+\sum_{\mu=1}^{2n}R_\mu\dot{\xi}_\mu+\dot{G}=0.\end{aligned} \tag{21}$$

Suppose that the system has a known integral

$$I=I(t,\boldsymbol{a})=\text{const.} \tag{22}$$

Firstly, differentiating it with respect to t, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+\sum_{\mu=1}^{2n}\frac{\partial I}{\partial a^\mu}\dot{a}^\mu=0. \tag{23}$$

Multiplying Eqs.(2) by

$$\overline{\xi}_\mu = \xi_\mu - \dot{a}^\mu \xi_0, \tag{24}$$

and carrying out the sum for s, we obtain

$$\sum_{\mu=1}^{2n} \xi_\mu^- \left(\sum_{\mu=1}^{2n} \Omega_{\mu\nu} \dot{a}^\nu - \frac{\partial B}{\partial a^\mu} - \frac{\partial R_\mu}{\partial t} - P_\mu \right) = 0. \tag{25}$$

Adding Eq.(23) to Eq.(25), separating the terms containing $\dot{a}^\nu$ and taking their coefficients as zeros, we obtain

$$\frac{\partial I}{\partial a^\nu} + \sum_{\mu=1}^{2n} \Omega_{\mu\nu} \xi_\mu + \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right) \xi_0 = 0,$$

$$(\nu = 1, 2, \cdots, 2n). \tag{26}$$

Letting

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}, \tag{27}$$

from Eqs.(26), we can obtain

$$\xi_\mu = \sum_{\nu=1}^{2n} \Omega^{\mu\nu} \left\{ \frac{\partial I}{\partial a^\nu} + \left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} + P_\nu \right) \xi_0 \right\},$$

$$(\mu = 1, 2, \cdots, 2n), \tag{28}$$

where

$$\sum_{\nu=1}^{2n} \Omega^{\mu\nu} \Omega_{\nu\rho} = \delta_{\mu\rho},$$

$$(\mu, \rho = 1, 2, \cdots, 2n). \tag{29}$$

We let the integral (22) equal the conserved quantity (19), i.e.

$$\sum_{\mu=1}^{2n} R_\mu \xi_\mu - B\xi_0 + G = I. \tag{30}$$

Thus we can seek the generators ξ_0 and ξ_μ from formulae (28) and (30) when the gauge function G is given, and the generators correspond to the Noether generalized quasi-symmetrical transformations of the corresponding free Birkhoff systems. If the generators satisfy the restriction conditions (20), then they are the Noether quasi-symmetrical transformations of Birkhoff systems with unilateral constraints.

Secondly, we must verify whether the Noether symmetries obtained above are Lie symmetries, i.e. by verifying whether the generators given by Eqs.(28) and (30) satisfy the determining equations (11)′, (16)′ and the restriction conditions (12)′.

Hence, we have

Proposition 3: If the integral (22) of the Birkhoff systems with unilateral constraints (1) and (2) is given, then the infinitesimal group transformations determined by Eqs.(28) and (30) must be the strong Lie symmetrical transformations, once they satisfy the determining equations (11)′, (16)′ and restriction conditions (12)′.

4. Example

We consider a fourth-order Birkhoff system

$$R_1 = a^3, \ R_2 = a^4, \ R_3 = R_4 = 0,$$

$$B = \frac{1}{2}\{(a^1)^2 + (a^3)^2 + (a^4)^2\}. \tag{31}$$

The motion of the system is subjected to the unilateral constraints

$$f_1 = a^1 a^3 - c_1^2 \geq 0,$$

$$f_2 = a^1 + a^4 - c_2 \geq 0. \tag{32}$$

We study the Lie symmetries and the conserved quantities of this system.

Firstly, we study the direct problem. From formula (31), we can obtain

$$(\Omega_{\mu\nu}) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$(\Omega^{\mu\nu}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \tag{33}$$

The differential equations of motion of the system can be expressed as:

$$-\dot{a}^3 - a^1 = \lambda_1 a^3 + \lambda_2,$$

$$-\dot{a}^4 = 0,$$

$$\dot{a}^1 - a^3 = \lambda_1 a^1,$$

$$\dot{a}^2 - a^4 = \lambda_2,$$

$$\lambda_1 \geq 0, \ \ f_2 \geq 0, \ \ \lambda_1 f_1 = 0,$$

$$\lambda_2 \geq 0, \ \ f_2 \geq 0, \ \ \lambda_2 f_2 = 0. \tag{34}$$

If the system is in the constrained state, i.e.

$$f_1 = a^1 a^3 - c_1^2 = 0,$$

$$f_2 = a^1 + a^4 - c_2 = 0, \tag{35}$$

then substituting Eqs.(34) into Eqs.(35), we obtain

$$\lambda_1 = -\frac{a^3}{a^1}, \lambda_2 = -a^1 + \frac{(a^3)^2}{a^1}. \tag{36}$$

Thus we have

$$P_1 = \lambda_1 \frac{\partial f_1}{\partial a^1} + \lambda_2 \frac{\partial f_2}{\partial a^2} = \lambda_1 a^3 + \lambda_2 = -a^1,$$

$$P_2 = 0, \quad P_3 = -a^3,$$

$$P_4 = -a^1 + \frac{(a^3)^2}{a^1}. \tag{37}$$

Equations (34) become

$$\dot{a}^1 = 0, \quad \dot{a}^2 = -a^1 + a^4 + \frac{(a^3)^2}{a^1},$$

$$\dot{a}^3 = 0, \quad \dot{a}^4 = 0. \tag{38}$$

If the system is divorced from the constraints, i.e.

$$f_1 = a^1 a^3 - c_1^2 > 0,$$

$$f_2 = a^1 + a^4 - c_2 > 0, \tag{39}$$

we obtain $\lambda_1 = \lambda_2 = 0$, and Eqs.(34) become

$$\dot{a}^1 = a^3, \quad \dot{a}^2 = a^4,$$

$$\dot{a}^3 = -a^1, \quad \dot{a}^4 = 0. \tag{40}$$

The determining equations $(11)'$ and $(16)'$ give

$$\begin{cases} \dot{\xi}_1 = 0, \\ \dot{\xi}_2 - \left[a^4 - a^1 + \dfrac{(a^3)^2}{a^1}\right]\dot{\xi}_0 \\ = \xi_4 - \xi_1 - \dfrac{(a^3)^2}{(a^1)^2}\xi_1 + \dfrac{2a^3}{a^1}\xi_3, \\ \dot{\xi}_3 = 0, \qquad \text{for} \begin{cases} f_1 = 0, \\ f_2 = 0, \end{cases} \\ \dot{\xi}_4 = 0, \end{cases} \tag{41}$$

$$\begin{cases} \dot{\xi}_1 - a^3\dot{\xi}_0 = \xi_3, \\ \dot{\xi}_2 - a^4\dot{\xi}_0 = \xi_4, \\ \dot{\xi}_3 + a^1\dot{\xi}_0 = -\xi_1, \quad \text{for} \begin{cases} f_1 > 0, \\ f_2 > 0, \end{cases} \\ \dot{\xi}_4 = 0, \end{cases} \tag{42}$$

and the restriction conditions $(12)'$ give

$$\begin{aligned} &a^3\xi^1 + a^1\xi_3 = 0, \\ &\xi_1 + \xi_4 = 0, \end{aligned} \quad \text{for} \begin{cases} f_1 = 0, \\ f_2 = 0. \end{cases} \tag{43}$$

Let the infinitesimal transformation be

$$\xi_0 = 0, \quad \xi_1 = 0, \quad \xi_2 = 1, \quad \xi_3 = 0, \quad \xi_4 = 0. \tag{44}$$

Obviously, formulae (41), (42) and (43) can all be satisfied, therefore formula (44) corresponds to the Lie symmetrical transformation. Substituting Eq.(44) into the structure equations (17) and (18), we obtain the gauge function:

$$G = 0. \tag{45}$$

Substituting Eqs.(44) and (45) into Eq.(19), we obtain the conserved quantity of the system

$$I = a^4 = \text{const.} \tag{46}$$

Secondly, we study the inverse problem. Supposing that the system has an integral (46), we seek the corresponding Lie symmetry. The formulae (28) and (30) give

$$\xi_1 = 0,$$

$$\xi_2 = 1 + a^4 - a^1 + \left[\frac{(a^3)^2}{a^1}\right]\xi_0,$$

$$\xi_3 = 0,$$

$$\xi_4 = 0,$$

$$a^4\left[a^4 - a^1 + \frac{(a^3)^2}{a^1}\right]\xi_0 - B\xi_0 + G = 0. \tag{47}$$

Obviously, the generators given by Eqs.(47) satisfy the conditions (20), therefore they corresponds to the Noether symmetrical transformations of the systems (31) and (32). In particular, taking the following gauge function

$$G = 0, \tag{48}$$

we have

$$\xi_0 = 0, \quad \xi_1 = 0, \quad \xi_2 = 1, \quad \xi_3 = 0, \quad \xi_4 = 0. \tag{49}$$

It is easy to verify that, for the gauge function G given by Eq.(48), the generators given by Eqs.(49) satisfy the determining equations $(11)'$, $(16)'$ and the restriction conditions $(12)'$. Therefore, the generators (49) correspond to the strong Lie symmetrical transformation of Birkhoff systems with unilateral constraints under study.

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