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Integrating factors and conservation theorems for Hamilton's canonical equations of motion of variable mass nonholonomic nonconservative dynamical systems*

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We present a general approach to the construction of conservation laws for variable mass nonholonomic nonconservative systems. First, we give the definition of integrating factors, and we study in detail the necessary conditions for the existence of the conserved quantities. Then, we establish the conservation theorem and its inverse theorem for Hamilton's canonical equations of motion of variable mass nonholonomic nonconservative dynamical systems. Finally, we give an example to illustrate the application of the results.

Keywords: variable mass, nonholonomic nonconservative system, Hamilton's canonical equations, integrating factor, conservation law

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1. Introduction

With the development of modern science and new technology, the mechanical theory of variable mass has been widely used in the fields of jets, rockets and aerospace. As a result, there has been a drastic increase in the interest in research works on the dynamics of variable mass systems. In particular, the existence of a certain conserved quantity will help us to understand the partial physical state and quality of the systems studied when it is hard to solve the differential equations of motion of the system. Furthermore, it has become one of the most important directions in modern analytical mechanics. There are two main methods to seek the conserved quantity. One is based on the theory of the invariance of Hamilton's action under infinitesimal transformation, namely, the Noether theory.[1] The other method was given by Lutzky[2] in 1979 who introduced Lie's method of the expansive group of invariance for ordinary differential equations under infinitesimal transformations to mechanical systems and presented the Lie symmetries of differential equations of motion. Significant achievements have been obtained in the study of this method in recent years.[3−10]

In this paper, we seek the conserved quantity of the variable mass nonholonomic systems on the basis of the theory of integrating factors.[11] Compared with the two famous methods mentioned above, this method has its benefits in fewer conditions, easy calculation and wide practical applications. Also, it has a favourable prospect in the integration theory of analytical mechanics. Firstly, we give the definition of integrating factors for differential equations of motion and we study the necessary conditions for the existence of conserved quantities. Secondly, we establish the conservation theorem and its inverse theorem for Hamilton's canonical equations of motion of variable mass nonholonomic nonconservative dynamical systems. Finally, we present an example to illustrate the application of the results.

2. Hamilton's canonical equations of motion

Suppose a system consists of N particles. At the moment t, the mass of the particle i is supposed to be $m_i (i = 1, 2, \cdots, N)$. At the moment $t + \mathrm{d}t$, the mass of a small particle separated from the particle i or combined with the particle i is supposed to be $\mathrm{d}m_i$. The configuration of the system is determined by n generalized coordinates $q_s (s = 1, 2, \cdots, n)$ and

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the mass of the particle depends on time, generalized coordinate and generalized velocity

$$m_i = m_i(t, q, \dot{q}), \quad (i = 1, 2, \cdots, N). \tag{1}$$

The motion of the system is subject to g ideal bilateral nonholonomic constraints of Chetaev's type

$$f_\beta(t, q, \dot{q}) = 0, \quad (\beta = 1, 2, \cdots, g). \tag{2}$$

Limitations of the nonholonomic constraints (2) added on the virtual displacements δq_s are shown as Appell–Chetaev conditions

$$\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0, \quad (s = 1, 2, \cdots, n). \tag{3}$$

According to the principle of D'Alembert–Lagrange and the virtual displacements equations (3), and with the help of the usual Lagrange multiplier method, we have the equations of motion of a variable mass nonholonomic nonconservative system as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + P_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \quad (s = 1, 2, \cdots, n). \tag{4}$$

Here L is the Lagrange function of the system, λ_β are the constrained multipliers, Q_s are the nonpotential generalized forces and P_s are the generalized counter thrusts. According to Ref.[7], we have

$$\begin{aligned} P_s =& (\overline{R}_i + \dot{m}_i \dot{\overline{r}}_i) \cdot \frac{\partial \overline{r}_i}{\partial q_s} - \frac{1}{2}\dot{\overline{r}}_i \cdot \dot{\overline{r}}_i \frac{\partial m_i}{\partial q_s} \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\overline{r}}_i \cdot \dot{\overline{r}}_i \frac{\partial m_i}{\partial \dot{q}_s}\right) \\ & (s = 1, 2, \cdots, n), \end{aligned} \tag{5}$$

where $\overline{r}_i$ and $\dot{\overline{r}}_i$ are the vector radius and the velocity of the particle i respectively, and

$$\overline{R}_i = \frac{\mathrm{d}m_i}{\mathrm{d}t}\overline{u}_i, \tag{6}$$

with $\overline{u}_i$ being the relative velocity of the small particle corresponding to the particle i.

Before we integrate the differential equations of motion, we obtain λ_β from Eqs.(2) and (4), and express it as a function of t, q and $\dot{q}$. Thus, Eqs.(4) can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + P_s + \Lambda_s, \quad (s = 1, 2, \cdots, n), \tag{7}$$

where

$$\Lambda_s = \Lambda_s(t, q, \dot{q}) = \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s} \tag{8}$$

are the generalized nonholonomic reactive forces of the constraint.

Equations (7) are called differential equations of motion of holonomic systems corresponding to the variable mass nonholonomic systems (2) and (4).

Next, we introduce Hamilton's function as

$$H(t, q, p) = p_s \dot{q}_s - L. \tag{9}$$

From Eqs.(7) and (9), we have

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \tag{10}$$

$$\dot{p}_s = -\frac{\partial H}{\partial q_s} + \tilde{Q}_s + \tilde{P}_s + \tilde{\Lambda}_s, \quad (s = 1, 2, \cdots, n). \tag{11}$$

Equations (10) and (11) are called Hamilton's canonical equations of motion of holonomic systems corresponding to the variable mass nonholonomic systems (2) and (4).

3. Integrating factors and conservation theorems

3.1. Integrating factors

The invariant

$$\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{P}_s - \tilde{\Lambda}_s\right) G_s$$

reduces identically to

$$\begin{aligned} &\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{P}_s - \tilde{\Lambda}_s\right) G_s \\ \equiv& \frac{\mathrm{d}}{\mathrm{d}t}(p_s G_s - HR - B) \\ &+ \mu_s\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{P}_s - \tilde{\Lambda}_s\right), \end{aligned} \tag{12}$$

where R, B and μ_s are functions of time, generalized coordinates and generalized momenta. In this case, functions $G_s = G_s(t, q, p)$ are called integrating factors of the equations of motion (11).

Combining Eqs.(11) and (12), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(p_s G_s - HR - B) \\ =& -\mu_s\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{P}_s - \tilde{\Lambda}_s\right), \end{aligned} \tag{13}$$

and the following theorem can be obtained.

3.2. Conservation theorems

Theorem 1. If functions G_s are integrating factors of Eqs.(11), then there exists a conserved quantity (first integration)

$$D = p_s G_s - HR - B \tag{14}$$

of the holonomic systems (10) and (11), corresponding to the variable mass nonholonomic systems (2) and (4).

For a given dynamical system, if functions G_s are integrating factors of Eqs.(11), the necessary condition (13) must be satisfied for every set of functions G_s, R, B and μ_s. The condition can be written as

$$\begin{aligned} &- R\frac{\partial H}{\partial t} + (\tilde{Q}_s + \tilde{P}_s + \tilde{\Lambda}_s)\left(G_s - R\frac{\partial H}{\partial p_s}\right) \\ &+ p_s\dot{G}_s - \frac{\partial H}{\partial q_s}G_s - H\dot{R} - \dot{B} + I = 0, \end{aligned} \tag{15}$$

where we have used Eqs.(10) and (11) to calculate $\dot{H}$ and

$$I = \mu_s\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{P}_s - \tilde{\Lambda}_s\right). \tag{16}$$

If the set of functions G_s, R, B and μ_s satisfy the necessary condition (12), and the right-hand side of Eq.(14) is reduced to a constant, then we have a singular set of functions G_s, R, B and μ_s.

From these considerations and Theorem l, we can formulate the following theorem, which is convenient for finding first integrals of the equations of motion (10) and (11).

Theorem 2. For every nonsingular set of functions G_s, R, B and μ_s, which satisfies the necessary condition (15), there exists a conserved quantity (14) of the holonomic systems corresponding to the variable mass nonholonomic systems.

By integration of Eq.(15) the nonsingular set of functions G_s, R, B and μ_s, which gives a conserved quantity, can be obtained. In any case, if we have any solution of Eq.(15), which does not contain any integration constant, we call the solution a functional solution of Eq.(15). When a nonsingular functional solution of Eq.(15) is substituted into Eq.(14), we obtain an ordinary first integral of the dynamical system. Here, the only integration constant is the value of D. Vujanovic[12] has shown that, by a complete solution of Eq.(15) which contains a sufficient number of integration constants, we can finally form a finite solution of the equations of motion (10) and (11).

Obviously, in Eq.(15) $\dfrac{\mathrm{d}(\ )}{\mathrm{d}t}$ must be interpreted as the following operator

$$\frac{\mathrm{d}(\ )}{\mathrm{d}t} = \frac{\partial(\ )}{\mathrm{d}t} + \frac{\partial H}{\partial p_s}\frac{\partial(\ )}{\partial q_s} + \dot{p}_s\frac{\partial(\ )}{\partial p_s}. \tag{17}$$

4. Killing equations

It is important to find the functions $R = R(t,q,p)$, $B = B(t,q,p)$ and $G_s = G_s(t,q,p)$ while using the theory of integrating factors to seek the conserved quantity of a dynamical system. Usually, we expand the necessary condition (15) and split it into first-order partial differential equations for G_s, R and B. This kind of partial differential equation is called the Killing equations. We can obtain this kind of function through solving Killing equations.

1. We can expand Eq.(15) using Eqs.(16) and (17), and divide it into terms with and without $\dot{p}_\beta$, and set them equal to zero separately. Then we have

$$\begin{aligned} &- R\frac{\partial H}{\partial t} + (\tilde{Q}_s + \tilde{P}_s + \tilde{\Lambda}_s)\left(G_s - R\frac{\partial H}{\partial p_s}\right) \\ &+ p_s\left(\frac{\partial G_s}{\partial t} + \frac{\partial G_s}{\partial q_\beta}\frac{\partial H}{\partial p_\beta}\right) - \frac{\partial H}{\partial q_s}G_s \\ &- H\left(\frac{\partial R}{\partial t} + \frac{\partial R}{\partial q_\beta}\frac{\partial H}{\partial p_\beta}\right) - \frac{\partial B}{\partial t} - \frac{\partial B}{\partial q_\beta}\frac{\partial H}{\partial p_\beta} \\ &+ \mu_\beta\left(\frac{\partial H}{\partial q_\beta} - \tilde{Q}_\beta - \tilde{P}_\beta - \tilde{\Lambda}_\beta\right) = 0, \end{aligned} \tag{18}$$

$$p_s\frac{\partial G_s}{\partial p_\beta} - H\frac{\partial R}{\partial p_\beta} - \frac{\partial B}{\partial p_\beta} + \mu_\beta = 0. \tag{19}$$

If the system admits a nonsingular solution with respect to $R = R(t,q,p)$, $B = B(t,q,p)$, $G_s = G_s(t,q,p)$ and $\mu_s = \mu_s(t,q,p)$, then a first integral (14) exists automatically. Here, we have a system of $(n+1)$ linear partial differential equations with, in most general cases, $(2n+2)$ unknown functions. Thus, there is a great deal of freedom in their solution. The most logical suggestion is to prescribe $(n+1)$ functions in advance and, after that, to solve the equations with respect to the remaining $(n+1)$ unknowns. Usually, the prescribed functions are B and μ_s.

2. We can expand the necessary condition (15) using Eqs.(11), (16) and (17), and obtain

$$\begin{aligned} &- R\frac{\partial H}{\partial t} + (\tilde{Q}_s + \tilde{P}_s + \tilde{\Lambda}_s)\left(G_s - R\frac{\partial H}{\partial p_s}\right) \\ &+ p_s\left(\frac{\partial G_s}{\partial t} + \frac{\partial G_s}{\partial q_\beta}\frac{\partial H}{\partial p_\beta}\right) - \frac{\partial H}{\partial q_s}G_s \\ &- H\left(\frac{\partial R}{\partial t} + \frac{\partial R}{\partial q_\beta}\frac{\partial H}{\partial p_\beta}\right) - \frac{\partial B}{\partial t} - \frac{\partial B}{\partial q_\beta}\frac{\partial H}{\partial p_\beta} \\ &+ \left(\tilde{Q}_\beta + \tilde{P}_\beta + \tilde{\Lambda}_\beta - \frac{\partial H}{\partial q_\beta}\right) \\ &\times \left(p_s\frac{\partial G_s}{\partial p_\beta} - H\frac{\partial R}{\partial p_\beta} - \frac{\partial B}{\partial p_\beta}\right) = 0. \end{aligned} \tag{20}$$

Equation (20) is a linear partial differential equation. In the general case, each of the $(n+2)$ functions G_s, R and B can be regarded as unknown. According to Theorem 2, any nonsingular solution of Eq.(20) with respect to G_s, R and B yields a first integral (14).

5. Inverse theorems of theorems 1 and 2

The connection between the first integral of the dynamical systems, based on the theory of integration, and the corresponding integrating factors G_s and functions R and B must be consistent with the necessary condition (15). It is shown that the necessary condition (15) is equivalent to Eqs.(18), (19) or (20). Therefore, the set of $(n+2)$ functions G_s, R and B must satisfy Eqs.(14), (18) and (19) or (14) and (20).

Calculating $\dfrac{\partial B}{\partial p_\beta}$ from Eq.(14), and forcing this result to satisfy Eqs.(19), we have

$$G_s = \frac{\partial D}{\partial p_s} + R\frac{\partial H}{\partial p_s}. \tag{21}$$

Now, combining Eqs.(14) and (21), we obtain

$$B = p_s\frac{\partial D}{\partial p_s} - D + R\left(p_s\frac{\partial H}{\partial p_s} - H\right). \tag{22}$$

Next, letting the first integral $D = D(t,q,p)$ equal the conserved quantity, we have

$$p_s G_s - HR - B = D(t,q,p). \tag{23}$$

From Eqs.(21), (22) and (23), G_s, R and B can be obtained.

In this way, there is a Theorem as follows.

Theorem 3. If every constant of the motion $D(t,q,p)$ for a dynamical system can be described by a Hamiltonian H, nonconservative generalized forces $\tilde{Q}_s$, generalized counter thrusts $\tilde{P}_s$ and generalized constraint reaction forces $\tilde{\Lambda}_s$, the integrating factors G_s and functions R and B corresponding to D are determined by Eqs.(21), (22) and (23).

6. Example

A particle of variable mass is studied, whose mass is given by

$$m = m_0\mathrm{e}^{-at} \quad (m_0 = \text{const},\ a = \text{const},\ a > 0). \tag{24}$$

The Lagrange function of the particle is

$$L = \frac{1}{2}m(t)(\dot{q}_1^2 + \dot{q}_2^2), \tag{25}$$

and the nonpotential generalized forces are

$$\left.\begin{aligned} Q_1 &= -\frac{1}{t(1+t^2)}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1),\\ Q_s &= -\frac{1}{1+t^2}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1). \end{aligned}\right\} \tag{26}$$

The separable absolute velocity of the particle is zero. Thus, we have

$$\overline{u} = -\dot{\overline{r}} = -\dot{q}_1\overline{i} - \dot{q}_2\overline{j}. \tag{27}$$

The nonholonomic constraint of the system is

$$f = \dot{q}_2 - t\dot{q}_1 = 0. \tag{28}$$

We try to find the conserved quantities of the system.

Firstly, we study the direct problem to obtain the conserved quantities of the system.

The first step is to establish Hamilton canonical equations (10) and (11) corresponding to the holonomic system.

From Eqs.(24), (27) and (5), we have

$$P_1 = P_2 = 0. \tag{29}$$

From Eqs.(4), we obtain

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_1) &= -\frac{1}{t(1+t^2)}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1) - \lambda t,\\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_2) &= -\frac{1}{1+t^2}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1) - \lambda. \end{aligned}\right\} \tag{30}$$

From Eqs.(28) and (30), we have

$$\lambda = \frac{1}{1+t^2}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1). \tag{31}$$

Therefore, we have

$$\left.\begin{aligned} &Q_1 + P_1 + \Lambda_1\\ &= -\frac{1}{t(1+t^2)}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1),\\ &Q_2 + P_2 + \Lambda_2 = 0, \end{aligned}\right\} \tag{32}$$

$$\dot{q}_1 = \frac{p_1}{m}, \quad \dot{q}_2 = \frac{p_2}{m}. \tag{33}$$

The Hamilton function is

$$H = p_1\dot{q}_1 + p_2\dot{q}_2 - L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2), \tag{34}$$

$$\left.\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial q_1} + Q_1 + P_1 + \Lambda_1\\ &= -\frac{1}{t(1+t^2)}(m\dot{q}_1 + \dot{m}\dot{q}_2 - \dot{m}t\dot{q}_1),\\ \dot{p}_2 &= -\frac{\partial H}{\partial q_2} + Q_2 + P_2 + \Lambda_2 = 0. \end{aligned}\right\} \tag{35}$$

The second step is to establish the Killing equation.

From Eqs.(20), we have

$$\begin{aligned}
&\frac{1}{2}Ram(p_1^2+p_2^2)-\frac{1+t}{t(1+t^2)}(m\dot{q}_1+\dot{m}\dot{q}_2-\dot{m}t\dot{q}_1)\\
&\times(G_1-Rmp_1)+p_1\left(\frac{\partial G_1}{\partial t}+\frac{\partial G_1}{\partial q_1}\frac{p_1}{m}+\frac{\partial G_1}{\partial q_2}\frac{p_2}{m}\right)\\
&+p_2\left(\frac{\partial G_2}{\partial t}+\frac{\partial G_2}{\partial q_1}\frac{p_1}{m}+\frac{\partial G_2}{\partial q_2}\frac{p_2}{m}\right)\\
&-\frac{1}{2}m(p_1^2+p_2^2)\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial q_1}\frac{p_1}{m}+\frac{\partial R}{\partial q_2}\frac{p_2}{m}\right)\\
&-\frac{\partial B}{\partial t}-\frac{\partial B}{\partial q_1}\frac{p_1}{m}-\frac{\partial B}{\partial q_2}\frac{p_2}{m}\\
&+\left[-\frac{1+t}{t(1+t^2)}(m\dot{q}_1+\dot{m}\dot{q}_2-\dot{m}t\dot{q}_1)\right]\\
&\times\left[p_1\frac{\partial G_1}{\partial p_1}+p_2\frac{\partial G_2}{\partial p_1}-\frac{1}{2}m(p_1^2+p_2^2)\frac{\partial R}{\partial p_1}-\frac{\partial B}{\partial p_1}\right]=0
\end{aligned}\tag{36}$$

The third step is to seek the conserved quantity of the system.

The solution of Eq.(36) is

$$R=0,\quad G_1=0,\quad G_2=1,\quad B=0.\tag{37}$$

For the integrating factors G_1 and G_2 and the functions R and B, all satisfy the necessary condition (15) and, according to Theorem 1, the conserved quantity (14) can be given as

$$D=m\dot{q}_2=\text{const.}\tag{38}$$

Secondly, we study the inverse problem, in which integrating factors G_s and functions R and B can be obtained from the known first integral.

Suppose the system studied has a first integral

$$D=m\dot{q}_2=p_2=\text{const.}\tag{39}$$

From Eqs.(21) and (22), we have

$$\left.\begin{aligned}&G_1=0,\ G_2=1+Rmp_2\\&B=\frac{1}{2}Rm(p_1^2+p_2^2),\end{aligned}\right\}.\tag{40}$$

Therefore, we obtain

$$R=\frac{2B}{m(p_1^2+p_2^2)}.\tag{41}$$

When B =0, we have.

$$R=0,\quad G_1=0,\quad G_2=1.\tag{42}$$

These values are consistent with the results in Ref.[9].

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