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Bound states of the Klein–Gordon and Dirac equations for scalar and vector harmonic oscillator potentials

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We give the exact bound states of the Klein–Gordon and Dirac equations with equal scalar and vector harmonic oscillator potentials.

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1. Introduction

The harmonic oscillator is one of the most important models in classical physics and quantum physics, and is widely used in many branches of physics. In non-relativistic quantum mechanics, many authors have adequately studied the problem of the harmonic oscillator. But, in relativistic quantum mechanics, it has seldom been investigated. When a particle is in a strong potential field, the relativistic effect must be considered. Although this problem has been investigated in Refs.[1] and [2], the exact solution of a relativistic harmonic oscillator has not yet been given. In recent years, some authors have assumed that the scalar potential is equal to the vector potential and have solved the Klein-Gordon equation as well as the Dirac equation. Then they obtained the s-wave bound-state solution of particles in some typical potential fields.[3–6] In this paper, we also assume that the scalar potential and vector potential are equal to each other, we solve the Klein–Gordon and Dirac equations of a harmonic oscillator, and we present the exact solution of a relativistic harmonic oscillator.

2. Bound-state solution of the Klein–Gordon equation with scalar and vector potentials equal to each other

According to Ref.[7], the Klein–Gordon equation for a harmonic oscillator is ($\hbar = c = 1$)

$$p^2 - (E - kr^2/4)^2 = -(m + kr^2/4)^2, \tag{1}$$

where p is the momentum operator, E and m are the energy and rest mass of the harmonic oscillator respectively, and k is the elastic coefficient. From Eq.(1) we have the radial part of the Klein–Gordon equation of the three-dimensional isotropic relativistic harmonic oscillator

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rR(r)) - \left[\frac{l(l+1)}{r^2} + (m^2 - E^2) + \frac{1}{2}(m+E)kr^2\right]R(r) = 0. \tag{2}$$

To solve Eq.(2), letting

$$\alpha = (3+2l)/8, \quad \beta = \sqrt{k(E+m)}, \tag{3}$$

$$\rho = \frac{\beta}{\sqrt{2}}r^2, \quad R(r) = \mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2} r^l f(\rho), \tag{4}$$

and substituting Eqs.(3) and (4) into Eq.(2), we obtain a confluent hypergeometric equation

$$\rho f''(\rho) + (4\alpha - \rho)f'(\rho) - [2\alpha - \sqrt{2}(E-m)\beta/(4k)]f(\rho) = 0, \tag{5}$$

the solution of which is

$$f(\rho) = F[2\alpha - \sqrt{2}(E-m)\beta/(4k), 4\alpha, \rho], \tag{6}$$

where F represents the confluent hypergeometric function $F[\alpha, \gamma, z]$.[8] Combining Eqs.(4) and (6) gives the non-normalized radial wavefunction

$$R(r) = \mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2} r^l F\left[2\alpha - \frac{(E-m)\sqrt{2}\beta}{4k}, 4\alpha, \frac{\beta}{\sqrt{2}}r^2\right]. \tag{7}$$

For bound states, the confluent hypergeometric function $F[\alpha,\gamma,z]$ must terminate as a polynomial, which demands

$$2\alpha-\frac{(E-m)\sqrt{2}\beta}{4k}=-n \quad n=0,1,2,\cdots. \tag{8}$$

This gives the energy equation

$$(2l+3)\sqrt{k}-(E-m)\sqrt{2(E+m)}=-4n\sqrt{k}. \tag{9}$$

Equation (9) has one real root, giving the following energy eigenvalues

$$\begin{aligned} E_{n,l}&=(2m+8\cdot 2^{2/3}m^2A^{-1/3}+2^{1/3}A^{1/3})/6,\\ A&=-B+\sqrt{B^2-1024m^6},\\ B&=32m^3-27k(3+2l+4n)^2. \end{aligned} \tag{10}$$

Using the relationship of the confluent hypergeometric function with the Laguerre polynomial and the normalization condition of the Laguerre polynomial, we obtain the normalized radial wavefunction[8]

$$\begin{aligned} R_{n,l}(r)=&\frac{\sqrt{2}\beta^{2\alpha}}{2^{\alpha}\Gamma(4\alpha)}\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2}r^l\sqrt{\frac{\Gamma(n+4\alpha)}{n!}}\\ &\times F\left[-n,4\alpha,\frac{\beta}{\sqrt{2}}r^2\right]. \end{aligned} \tag{11}$$

3. Bound-state solution of Dirac equations with scalar and vector potentials equal to each other

The Dirac equation with scalar potential $S(r)$ and vector potential $V(r)$ is ($\hbar=c=1$)

$$\alpha\cdot\boldsymbol{p}+\beta[m+S(r)]\,\Psi=[E-V(r)]\,\Psi. \tag{12}$$

In relativistic quantum mechanics, the complete conservative quantity set of a particle in a central field can be taken to be (H,K,J^2,J_z), the eigenfunctions of which are[9]

$$\Psi=\frac{1}{r}\begin{bmatrix} F(r)\phi^A_{jm_j}\\ iG(r)\phi^B_{jm_j}\end{bmatrix} \quad K=j+1/2, \tag{13}$$

$$\Psi=\frac{1}{r}\begin{bmatrix} F(r)\phi^B_{jm_j}\\ iG(r)\phi^A_{jm_j}\end{bmatrix} \quad K=-(j+1/2), \tag{14}$$

where

$$\phi^A_{jm_j}=\frac{1}{\sqrt{2l+1}}\begin{bmatrix}\sqrt{l+m+1}Y_{lm}\\ \sqrt{l-m}Y_{lm+1}\end{bmatrix},$$

$$\phi^B_{jm_j}=\frac{1}{\sqrt{2l+3}}\begin{bmatrix}-\sqrt{l+1-m}Y_{l+1m}\\ \sqrt{l+1+m+1}Y_{l+1m+1}\end{bmatrix}. \tag{15}$$

Substituting Eqs.(13) or (14) into Eq.(12) gives the radial part of the Dirac equation

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}r}-\frac{K}{r}F&=[m+E+S(r)-V(r)]G,\\ \frac{\mathrm{d}G}{\mathrm{d}r}+\frac{K}{r}G&=[m-E+S(r)+V(r)]F. \end{aligned} \tag{16}$$

Taking $S(r)$=$V(r)$=$kr^2/4$, Eq.(16) becomes

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}r}-\frac{K}{r}F&=[m+E]G,\\ \frac{\mathrm{d}G}{\mathrm{d}r}+\frac{K}{r}G&=[m-E+kr^2/2]F. \end{aligned} \tag{17}$$

Eliminating G from Eq.(17) gives

$$\begin{aligned} &\frac{\mathrm{d}^2F}{\mathrm{d}r^2}-\frac{K(K-1)}{r^2}F\\ &+(E+m)(E-m-kr^2/2)F=0. \end{aligned} \tag{18}$$

Using the same procedure of solving Eq.(2), we have the solution of Eq.(18)

$$F(r)=\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2}r^Kf(\rho), \tag{19}$$

where

$$f(\rho)=F[2\alpha'-\sqrt{2}(E-m)\beta/(4k),4\alpha',\rho]. \tag{20}$$

β and ρ are the same as in Eqs.(3) and (4) respectively, but

$$\alpha'=(2K+1)/8. \tag{21}$$

The value which K can take ensures $\alpha'\neq 0,-1,-2,\cdots$. For bound states, the confluent hypergeometric function $F[\alpha,\gamma,z]$ must terminate as a polynomial, which means that

$$2\alpha'-\frac{(E-m)\sqrt{2}\beta}{4k}=-n \quad n=0,1,2,\cdots. \tag{22}$$

This gives the energy equation

$$(2K+1)\sqrt{k}-(E-m)\sqrt{2(E+m)}=-4n\sqrt{k}. \tag{23}$$

Equation (23) has one real root, which gives energy eigenvalues

$$\begin{aligned} E_{n,K}&=(2m+8\cdot 2^{2/3}m^2A^{-1/3}+2^{1/3}A^{1/3})/6,\\ A&=-B+\sqrt{B^2-1024m^6},\\ B&=32m^3-27k(1+2K+4n)^2. \end{aligned} \tag{24}$$

Substituting Eqs.(19) and (20) into Eq.(17), $G(r)$ can be obtained. According to the normalization condition

$$\int \Psi^+ \Psi \mathrm{d}r = \int (|F(r)|^2 + |G(r)|^2)\mathrm{d}r = 1 \qquad (25)$$

and by doing some tedious calculations, finally we have normalized $F(r)$ and $G(r)$

$$F(r) = C\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2} r^K F\left(-n, 4\alpha', -\frac{\beta}{\sqrt{2}}r^2\right), \qquad (26)$$

$$\begin{aligned} G(r) =& C\frac{\beta}{2\sqrt{2}\alpha'(m+E_{n,K})}\mathrm{e}^{-\frac{\beta}{2\sqrt{2}}r^2} r^{K+1} \\ & \times \left[nF\left(1-n, 4\alpha'+1, \frac{\beta}{\sqrt{2}}r^2\right)\right. \\ & \left. + 2\alpha' F\left(-n, 4\alpha', \frac{\beta}{\sqrt{2}}r^2\right)\right], \end{aligned} \qquad (27)$$

where the normalization constant is

$$C = \frac{2^{1-\alpha'}\beta^{2\alpha'}}{\Gamma(4\alpha')}\sqrt{\frac{\Gamma(n+4\alpha')(E_{n,K}+m)}{n!(3E_{n,K}+m)}}. \qquad (28)$$

Substituting $F(r)$ and $G(r)$ into Eqs.(13) and (14) gives the spinor wavefunction of the Dirac equation.

4. Uniform energy equation

Let

$$N = \begin{cases} 2n+l & \\ 2n+K-1 & \end{cases} \quad N = 0, 1, 2, \cdots \quad \begin{matrix} \text{in} & (9) \\ \text{in} & (23) \end{matrix} \qquad (29)$$

$$E' = E - m, \quad \omega = \sqrt{k/m}, \qquad (30)$$

then the energy equations (9) and (23) can be expressed uniformly as

$$\left(N + \frac{3}{2}\right)\omega - E'\sqrt{1 + \frac{E'}{2m}} = 0 \quad N = 0, 1, 2, \cdots, \qquad (31)$$

where ω is the proper frequency of a harmonic oscillator, and E' is the energy which does not include the rest energy of the particle. Expanding the square root in Eq.(31) as a series of E' and taking different powers of E', we obtain different approximate energy equations. Solving these equations gives different approximations of the relativistic harmonic oscillator energy.

5. Conclusion and problems

The above result shows that the Klein–Gordon and Dirac equations with equal scalar and vector potentials, which is equal to half of the harmonic oscillator potential, can be exactly solved. The exact wavefunction and energy levels of a relativistic harmonic oscillator can also be obtained.

Hou and Zhou[6] have also put forward two schemes for the relativistic harmonic oscillator. The first scheme takes the total harmonic oscillator potential as the scalar potential, and the second takes the total harmonic oscillator potential as the vector potential. Unfortunately, for these two situations, it is too difficult to solve the Klein–Gordon or Dirac equations exactly. This needs to be studied further.

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