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Bäcklund transformation and variable separation solutions for the generalized Nozhnik-Novikov-Veselov equation*

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Using the extended homogeneous balance method, the Bäcklund transformation for a (2+1)-dimensional integrable model, the generalized Nizhnik–Novikov–Veselov (GNNV) equation, is first obtained. Also, making use of the Bäcklund transformation, the GNNV equation is changed into three equations: linear, bilinear and trilinear form equations. Starting from these three equations, a rather general variable separation solution of the model is constructed. The abundant localized coherent structures of the model can be induced by the entrance of two variable-separated arbitrary functions.

Keywords: extended homogeneous balance method, (2+1) dimensions, GNNV equation, localized coherent structures

PACC: 0290, 0340, 0365

1. Introduction

Recently, two types of "variable separating" procedure have been established, which are extensions of the variable separation method in the linear case. The first type is called the symmetry constraint or nonlinearization of the Lax pairs,[1] because this type of procedure is used only for integrable models which possess Lax pairs. Lou and Chen[2] extended the method to any models which may possess Lax pairs or not, so the method is equivalently called the "formal variable separation approach" (FVSA). The independent variables of a reduced field in the FVSA have not been totally separated though the reduced field satisfies some lower-dimensional equations. The second type of variable separation method is established first for the (2+1)-dimensional Davey–Stewartson (DS) and asymmetric DS equations,[3] and then generalized to the (2+1)-dimensional Korteweg–de Vries (KdV) equation, the Nozhnik–Novikov–Veselov (NNV) equation, and the (2+1)-dimensional nonlinear Schrödinger equation.[4−6] The basic idea is that by solving its bilinear form and introducing a prior ansatz, some special types of exact solution of the (2+1)-dimensional nonlinear model can be obtained from two (1+1)-dimensional variable separation fields. In this paper, by using the extended homogeneous balance method,[15] we propose a new variable separation approach for solving (2+1)-dimensional nonlinear models.

It is well known that for the (2+1)-dimensional NNV equation[7]

$$u_t + au_{xxx} + bu_{yyy} - 3a(uv)_x - 3b(uw)_y = 0,$$

$$u_x = v_y, \qquad u_y = w_x,$$

which is only a known isotropic Lax extension of the well-known (1+1)-dimensional KdV equation. Some types of soliton solutions have been studied by many authors. For instance, Boiti *et al*[8] solved the NNV equation via the inverse scattering transformation. Tagami[9] and Hu and Li[10] obtained the soliton-like solutions of the NNV equation by means of the Bäcklund transformation. Hu[11] has also given the nonlinear superposition formula of the NNV equation. Ohta[12] obtained the Pfaffian solutions for the NNV equation. Radha and Lakshmanan[13] constructed only its dromion solutions from its bilinear form after analysing its integrability aspects. Lou obtained some new special types of multi-soliton solution[14] by using the standard truncated Painlevé analysis and

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the localized solutions by using the variable separation method[4] for the NNV equation.

But the following (2+1)-dimensional generalized NNV (GNNV) equation

$$u_t + au_{xxx} + bu_{yyy} + cu_x + du_y - 3a(uv)_x - 3b(uw)_y) = 0, \quad (1)$$

$$u_x = v_y, \qquad u_y = w_x, \quad (2)$$

has not yet been well studied. Boiti *et al*[8] solved the GNNV equation via the inverse scattering transformation, and Radha and Lakshmanan[13] constructed its multi-dromion solutions. In this paper, we study the GNNV equation using the variable separation approach based on the extended homogeneous balance method.[15]

In section 2, a general theory of the extended homogeneous balance method is described. In section 3, after using the extended homogeneous balance method, Bäcklund transformation of the GNNV equation is given. In section 4, a rather general variable separation solution for the model is obtained. Finally, section 5 contains a short discussion of the results.

2. General theory of extended homogeneous balance method

The homogeneous balance method has shown its efficiency in finding analytic solitary wave solutions of nonlinear models. Here we first describe the general theory of the extended homogeneous balance method. The n-dimensional nth-order partial differential equation is given by

$$P(x_0 \equiv t, x_1, x_2, \cdots, x_n, u, u_{x_i} u_{x_i x_j}, \cdots u_{x_{i_1} x_{i_2} \cdots x_{i_n}}) = 0, \quad (3)$$

where P is a polynomial function of the field u and its derivatives. Supposing $f = f(\phi)$ is a function of $\phi = \phi(x_0, x_1, x_2, \cdots x_n)$, we can construct the solution of Eq.(3) with a suitable linear combination of

$$1, f(\phi), f(\phi)_{x_i}, f(\phi)_{x_i x_j}, \cdots \quad (4)$$

The homogeneous balance method mainly consists of the following five steps:

(i) As the leading part of the analysis, we select a suitable linear combination of the functions in (4), its coefficients to be determined, such that the highest nonlinear terms and the highest-order partial derivative terms in Eq.(3) can be partially balanced.

(ii) Substituting the chosen combination into Eq.(3), Eq.(3) is transformed into polynomials with the same highest degree in partial derivatives of $\phi(x_0, x_1, \cdots, x_n)$ in spite of $f(\phi)$ and its derivatives of different degrees. Collecting all the terms with the highest degree of derivatives of $\phi(x_0, x_1, \cdots, x_n)$ and setting its coefficient to zero, an ordinary differential equation for $f(\phi)$ is obtained.

(iii) Starting from the ordinary differential equation of $f(\phi)$, not only the solution of $f(\phi)$ is obtained, but also the nonlinear terms of various derivatives of $f(\phi)$ in the expression obtained in the second step can be replaced by the corresponding higher-order derivatives of $f(\phi)$.

(iv) After some detailed calculation, we collect the terms with the same order of derivatives of $f(\phi)$. Setting the coefficient of each order of derivative of $f(\phi)$ to zero, a set of linear and bilinear, even trilinear, partial differential equations for function $\phi(x_0, x_1, \cdots, x_n)$ are obtained.

(v) By solving the equations obtained in the fourth step, we can find many solutions of $\phi(x_0, x_1, \cdots, x_n)$.

(vi) Making use of the results of $\phi(x_0, x_1, \cdots, x_n)$, we can construct the abundant exact solution structures of Eq.(3).

3. Bäcklund transformation for the GNNV equation

Now we use the homogeneous balance method to derive a nonlinear transformation of the GNNV equation. In order that the nonlinear terms and the higher-order partial derivative terms in systems (1) and (2) can be partially balanced, we suppose that their solutions are of the form

$$u = f(\varphi)_{xy} + u_0 = f''(\varphi)\varphi_x\varphi_y + f'(\varphi)\varphi_{xy} + u_0, \quad (5)$$

$$v = g(\varphi)_{xx} + v_0 = g''(\varphi)\varphi_x^2 + g'(\varphi)\varphi_{xx} + v_0, \quad (6)$$

$$w = h(\varphi)_{yy} + w_0 = h''(\varphi)\varphi_y^2 + h'(\varphi)\varphi_{yy} + w_0, \quad (7)$$

where $f(\varphi)$, $g(\varphi)$ and $h(\varphi)$ are functions of one argument φ only, $\varphi = \varphi(x, y, t)$, a prime denotes $\dfrac{\mathrm{d}}{\mathrm{d}\varphi}$, and $\{u_0, v_0, w_0\}$ is an arbitrary known seed solution of the GNNV equation.

From Eqs.(5)–(7), it is easy to deduce that

$$u_x = f^{(3)}\varphi_x^2\varphi_y + 2f''\varphi_x\varphi_{xy} + f''\varphi_{xx}\varphi_y + f'\varphi_{xxy} + u_{0x}, \quad (8)$$

$$u_y = f^{(3)}\varphi_y^2\varphi_x + 2f''\varphi_{xy}\varphi_y + f''\varphi_{yy}\varphi_x + f'\varphi_{xyy} + u_{0y}, \quad (9)$$

$$
\begin{aligned}
u_{xxx} =& f^{(5)}\varphi_x^4\varphi_y + 6f^{(4)}\varphi_x^2\varphi_{xx}\varphi_y + 4f^{(4)}\varphi_x^3\varphi_{xy} \\
&+ 12f^{(3)}\varphi_x\varphi_{xx}\varphi_{xy} + 6f^{(3)}\varphi_x^2\varphi_{xxy} \\
&+ 3f^{(3)}\varphi_{xx}^2\varphi_y + 4f^{(3)}\varphi_x\varphi_{xxx}\varphi_y \\
&+ 6f''\varphi_{xx}\varphi_{xxy} + 4f''\varphi_x\varphi_{xxxy} \\
&+ f''\varphi_{xxxx}\varphi_y + f'\varphi_{xxxxy} + u_{0xxxx}, \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
u_{yyy} =& f^{(5)}\varphi_y^4\varphi_x + 6f^{(4)}\varphi_y^2\varphi_{yy}\varphi_x + 4f^{(4)}\varphi_y^3\varphi_{xy} \\
&+ 12f^{(3)}\varphi_y\varphi_{yy}\varphi_{xy} + 6f^{(3)}\varphi_y^2\varphi_{yyx} \\
&+ 4f^{(3)}\varphi_y\varphi_{yyy}\varphi_x + 4f''\varphi_y\varphi_{yyyx} \\
&+ 3f^{(3)}\varphi_{yy}^2\varphi_x + f''\varphi_{yyyy}\varphi_x \\
&+ f'\varphi_{yyyyx} + u_{0yyy}, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
u_t =& f^{(3)}\varphi_x\varphi_y\varphi_t + f''(\varphi_{xt}\varphi_y + \varphi_x\varphi_{yt} \\
&+ \varphi_t\varphi_{xy}) + f'\varphi_{xyt} + u_{0t}, \qquad (12)
\end{aligned}
$$

$$
v_x = g^{(3)}\varphi_x^3 + 3g''\varphi_x\varphi_{xx} + g'\varphi_{xxx} + v_{0x}, \qquad (13)
$$

$$
\begin{aligned}
v_y =& g^{(3)}\varphi_x^2\varphi_y + 2g''\varphi_x\varphi_{xy} \\
&+ g''\varphi_{xx}\varphi_y + g'\varphi_{xxy} + v_{0y}, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
w_x =& h^{(3)}\varphi_y^2\varphi_x + 2h''\varphi_{xy}\varphi_y \\
&+ h''\varphi_x\varphi_{yy} + h'\varphi_{xyy} + w_{0x}, \qquad (15)
\end{aligned}
$$

$$
w_y = h^{(3)}\varphi_y^3 + 3h''\varphi_{yy}\varphi_y + h'\varphi_{yyy} + v_{0y}. \qquad (16)
$$

Substituting Eqs.(8)–(12) and (13)–(16) into Eqs.(1) and (2) respectively, and collecting all homogeneous terms in the partial derivatives of $\varphi(x, y, t)$, we have

$$
\begin{aligned}
&(f^{(5)} - 3f''g^{(3)} - 3f^{(3)}g'')a\varphi_x^4\varphi_y + (f^{(5)} - 3f''h^{(3)} - 3f^{(3)}h'')b\varphi_x\varphi_y^4 \\
&+ (6f^{(4)} - 12f''g'' - 3f^{(3)}g')a\varphi_x^2\varphi_{xx}\varphi_y + (6f^{(4)} - 12f''h'' - 3f^{(3)}h')b\varphi_x\varphi_y^2\varphi_{yy} \\
&+ (4f^{(4)} - 3f'g^{(3)} - 6f''g'')a\varphi_x^3\varphi_{xy} + (4f^{(4)} - 3f'h^{(3)} - 6f''g'')b\varphi_y^3\varphi_{xy} \\
&+ f^{(3)}(\varphi_x\varphi_y\varphi_t + 12a\varphi_x\varphi_{xx}\varphi_{xy} + 12b\varphi_y\varphi_{yy}\varphi_{xy} + 6a\varphi_x^2\varphi_{xxy} + 6b\varphi_y^2\varphi_{xyy} \\
&+ 4a\varphi_x\varphi_y\varphi_{xxx} + 4b\varphi_x\varphi_y\varphi_{yyyx} + 3a\varphi_{xx}^2\varphi_y + 3b\varphi_x\varphi_{yy}^2 + c\varphi_x^2\varphi_y + d\varphi_y^2\varphi_x \\
&- 3av_0\varphi_x^2\varphi_y - 3bw_0\varphi_y^2\varphi_x) - 3au_0g^{(3)}\varphi_x^3 - 3bu_0h^{(3)}\varphi_y^3 - 3a(3\varphi_x\varphi_{xx}\varphi_{xy} + \varphi_x^2\varphi_{xxy})f'g'' \\
&- 3b(3\varphi_y\varphi_{yy}\varphi_{xy} + \varphi_y^2\varphi_{xyy})f'h'' - 3a(\varphi_x\varphi_{xxx}\varphi_y + \varphi_{xx}^2\varphi_y + 2\varphi_x\varphi_{xx}\varphi_{xy})f''g' \\
&- 3b(\varphi_x\varphi_y\varphi_{yyy} + \varphi_x\varphi_{yy}^2 + 2\varphi_y\varphi_{xy}\varphi_{yy})f''h' - 3a(\varphi_{xy}\varphi_{xxx} + \varphi_{xx}\varphi_{xxy})f'h' \\
&- 3b(\varphi_{xy}\varphi_{yyy} + \varphi_{yy}\varphi_{xyy})f'g' + f''(\varphi_{xt}\varphi_y + \varphi_x\varphi_{yt} + \varphi_t\varphi_{xy} \\
&+ 4a\varphi_{xxx}\varphi_{xy} + 6a\varphi_{xx}\varphi_{xxy} + 4a\varphi_x\varphi_{xxxy} + 4b\varphi_{xy}\varphi_{yyy} + 6b\varphi_{yy}\varphi_{yyx} + 4b\varphi_y\varphi_{xyyy} \\
&+ a\varphi_{xxxx}\varphi_y + b\varphi_x\varphi_{yyyy}c\varphi_{xx}\varphi_y + d\varphi_{yy}\varphi_x + 2c\varphi_x\varphi_{xy} + 2d\varphi_y\varphi_{xy} - 6av_0\varphi_x\varphi_{xxy} \\
&- 3av_0\varphi_y\varphi_{xx} - 6bw_0\varphi_y\varphi_{xy} - 3bw_0\varphi_x\varphi_{yy} - 3av_{0x}\varphi_x\varphi_y - 3bw_{0y}\varphi_x\varphi_y) \\
&- 3ag''(3u_0\varphi_x\varphi_{xx} + u_{0x}\varphi_x^2) - 3bh''(3u_0\varphi_y\varphi_{yy} + u_{0y}\varphi_y^2) \\
&- 3bf'(\varphi_{xyt} + a\varphi_{xxxxy} + b\varphi_{xyyyy} - 3av_{0x}\varphi_{xy} + c\varphi_{xxy} + d\varphi_{xyy} - 3av_0\varphi_{xxy} - 3bw_{0y}\varphi_{xy} \\
&- 3bw_0\varphi_{xyy}) - 3ag'(u_0\varphi_{xxx} + u_{0x}\varphi_{xx}) - 3bh'(u_0\varphi_{yyy} + u_{0y}\varphi_{yy}) = 0, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
&(f^{(3)} - g^{(3)})\varphi_x^2\varphi_y + (f'' - g'')\varphi_{xx}\varphi_y \\
&+ 2(f'' - g'')\varphi_x\varphi_{xy} \\
&+ (f' - g')\varphi_{xxy} + u_{0x} - v_{0y} = 0, \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
&(f^{(3)} - h^{(3)})\varphi_x\varphi_y^2 + (f'' - h'')\varphi_x\varphi_{yy} \\
&+ 2(f'' - g'')\varphi_y\varphi_{xy} \\
&+ (f' - g')\varphi_{xyy} + u_{0y} - w_{0x} = 0. \qquad (19)
\end{aligned}
$$

For simplicity, from Eqs.(18) and (19), we take the special solution as

$$
f = g = h,\ u_0 = 0, v_0 = v_0(x, t), w_0 = w_0(y, t). \qquad (20)
$$

Now we can simplify Eq.(17). First setting the coefficients of the terms with $a\varphi_x^4\varphi_y$ and $b\varphi_y^4\varphi_x$ to zero, we obtain an ordinary differential equation for function $f(\varphi)$

$$
f^{(5)} - 6f'f^{(3)} = 0, \qquad (21)
$$

which indicates that the nonlinear terms and the second-order derivative terms appearing in Eqs.(1) and (2) have been partially balanced. This is why we assume that the solution of Eqs.(1) and (2) is of the form Eqs.(5)–(7). Solving Eq.(19), we have

$$
f = -2\ln\varphi, \qquad (22)
$$

which yields

$$f'^2 = 2f'', \quad f'f'' = f^{(3)},$$
$$f''^2 = \frac{1}{3}f^{(4)}, \quad f'f^{(3)} = \frac{2}{3}f^{(4)}. \tag{23}$$

By using Eqs.(21)–(23), Eq.(17) can be simplified as follows

$$\begin{aligned}
&f^{(3)}[\varphi_x\varphi_y(\varphi_t + a\varphi_{xxx} + b\varphi_{yyy} + c\varphi_x \\
&+ d\varphi_y - 3av_0\varphi_x - 3bw_0\varphi_y) \\
&+ 3(a\varphi_x\varphi_{xx}\varphi_{xy} + b\varphi_y\varphi_{yy}\varphi_{xy} \\
&- a\varphi_x^2\varphi_{xxy} - b\varphi_y^2\varphi_{xyy})] \\
&+ f''[\varphi_y(\varphi_t + a\varphi_{xxx} + b\varphi_{yyy} + c\varphi_x \\
&+ d\varphi_y - 3av_0\varphi_x - 3bw_0\varphi_y)_x \\
&+ \varphi_x(\varphi_t + a\varphi_{xxx} + b\varphi_{yyy} + c\varphi_x \\
&+ d\varphi_y - 3av_0\varphi_x - 3bw_0\varphi_y)_y \\
&+ \varphi_{xy}(\varphi_t + a\varphi_{xxx} + b\varphi_{yyy} + c\varphi_x + d\varphi_y \\
&- 3av_0\varphi_x - 3bw_0\varphi_y) + 3(-a\varphi_{xxx}\varphi_{xy} \\
&+ a\varphi_x\varphi_{xxxy} - b\varphi_{xy}\varphi_{yyy} + b\varphi_y\varphi_{xyyy})] \\
&+ f'(\varphi_t + a\varphi_{xxx} + b\varphi_{yyy} + c\varphi_x \\
&+ d\varphi_y - 3av_0\varphi_x - 3bw_0\varphi_y)_{xy} = 0.
\end{aligned} \tag{24}$$

Setting the coefficients of $f^{(3)}$, f'' and f' in Eq.(24) to zero yields a set of equations satisfied by $\varphi(x,y,t)$

$$\begin{aligned}
&\varphi_t + a\varphi_{xxx} + b\varphi_{yyy} + c\varphi_x \\
&+ d\varphi_y - 3av_0\varphi_x - 3bw_0\varphi_y = 0,
\end{aligned} \tag{25}$$

$$a\varphi_x\varphi_{xxxy} + b\varphi_y\varphi_{xyyy} - a\varphi_{xxx}\varphi_{xy} - b\varphi_{xy}\varphi_{yyy} = 0, \tag{26}$$

$$a\varphi_x\varphi_{xx}\varphi_{xy} + b\varphi_y\varphi_{yy}\varphi_{xy} - a\varphi_x^2\varphi_{xxy} - b\varphi_y^2\varphi_{xyy} = 0. \tag{27}$$

Thus it can be seen that, in terms of Eqs.(20)–(21), the GNNV equation may be simplified into Eqs.(25)–(27) by using the above transformations (5)–(7), that is

$$u = \frac{2\varphi_x\varphi_y}{\varphi^2} - \frac{2\varphi_{xy}}{\varphi}, \tag{28}$$

$$v = \frac{2\varphi_x^2}{\varphi^2} - \frac{2\varphi_{xx}}{\varphi} + v_0(x,t), \tag{29}$$

$$w = \frac{2\varphi_y^2}{\varphi^2} - \frac{2\varphi_{yy}}{\varphi} + w_0(y,t), \tag{30}$$

which are just the Bäcklund transformations of the GNNV equation.

4. Variable separation solution for the (2+1)-dimensional GNNV equation

Because $v_0(x,t)$ and $w_0(y,t)$ are arbitrary functions of variables $\{x,t\}$ and $\{y,t\}$ respectively, from Eqs.(25)–(27) we can find the following variable separation solution

$$\phi(x,y,t) = 1 + a_1p(x,t) + a_2q(q,t) + Ap(x,t)q(y,t),$$
$$v_0(x,t) = \frac{p_t + ap_{xxx} + cp_x}{3ap_x},$$
$$w_0(y,t) = \frac{q_t + bq_{yyy} + dq_v}{3aq_y}, \tag{31}$$

where $p(x,t)$ is an arbitrary function of variables $\{x,t\}$, $q,(y,t)$ is an arbitrary function of variable $\{y,t\}$, and a_1, a_2 and A are three arbitrary constants. Inserting solution (22) into Eq.(18), we obtain a rather general exact solitary wave solution of the (2+1)-dimensional GNNV equation in the form

$$u(x,y,t) = \frac{2(A - a_1a_2)p_xq_y}{[1 + a_1p + a_2q + Ap\,q]^2}, \tag{32}$$

$$\begin{aligned}
v(x,y,t) =& \frac{2(a_1 + Aq)^2p_x^2}{[1 + a_1p + a_2q + Ap\,q]^2} \\
&- \frac{2(a_1 + Aq)p_{xx}}{1 + a_1p + a_2q + Ap\,q} \\
&+ \frac{p_t + ap_{xxx} + cp_x}{3ap_x},
\end{aligned} \tag{33}$$

$$\begin{aligned}
w(x,y,t) =& \frac{2(a_2 + Ap)^2q_y^2}{[1 + a_1p + a_2q + Ap\,q]^2} \\
&- \frac{2(a_1 + Ap)q_{xx}}{1 + a_1p + a_2q + Ap\,q} \\
&+ \frac{q_t + bp_{xxx} + dq_y}{3aq_y}.
\end{aligned} \tag{34}$$

When $c = 0$ and $d = 0$, similar exact solutions to Eqs.(32)–(34) have been given by using the variable separation method in Ref.[4]. It is here obtained again by the extended homogeneous balance method. We can see that the calculation process herein is much simpler than in Ref.[4].

5. Summary and discussion

In summary, by using the extended homogeneous balance method for the (2+1)-dimensional GNNV equation, we may obtain a rather general variable separation solution because of the existence of two arbitrary functions. It has been pointed out[4] that the multiple localized structures may be dromions, lumps, ring solitons, breathers and instantons, etc, for the (2+1)-dimensional NNV equation by selecting the

arbitrary functions appropriately. It should be emphasized that the richness of the solitons caused by arbitrary functions may be a universal phenomenon in (2+1) and more dimensions. It can be seen from the above that there are more abundant coherent localized structures for the higher-dimensional integral models and even for some types of higher-dimensional nonintegrable models.

The variable separation method in linear physics is a very powerful method. Now, by the extended homogeneous balance method, we have obtained the variable separation solution to (2+1)-dimensional nonlinear integrable models. This fact describes that the extended homogeneous balance method is a very powerful method for the study of nonlinear physics model in higher dimensions. The applications of the method to other (2+1)-dimensional nonlinear physical models is worth discussing further.

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