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Solution to the Bargmann–Wigner equation for a half-integral spin*

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A rigorous method to solve the Bargmann–Wigner equation for an arbitrary half-integral spin is presented and explicit relativistic wavefunctions for an arbitrary half-integral spin are deduced.

Keywords: half-integral spin, Bargmann-Wigner equation, rigorous solution
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1. Introduction

The field theories that describe spin 0, 1/2 and 1 fields have been well established and they provide powerful tools in many areas of application. An extension of these to the theory that could describe fields of an arbitrary half-integral spin is of crucial importance both in theory and in applications. Theoretically, this theory is a foundation for the complete Fermi–Dirac statistics. In applications, the analysis of amplitudes in high-energy processes relies on explicit high-spin relativistic wavefunctions.[1−3]

Based on the work of Dirac and Fierz,[4−6] Bargmann and Wigner[7] found a set of general equations for an arbitrary spin more than half a century ago, to our knowledge. However, except for the simple cases of spin 1/2, 1 and 3/2, this set of equations has not been solved rigorously (see, for example, Ref.[8]). Recently, we have proposed a rigorous method to solve the Bargmann–Wigner (B–W) equation for spin 5/2.[9] In this paper, we generalize this method to the more general cases, that is to solve the Bargmann–Wigner equation for an arbitrary half-integral spin and to deduce correspondingly the explicit relativistic wavefunctions, and thus to constitute a complete theoretical system.

2. Bargmann–Wigner equation for an arbitrary half-integral spin

The original form of the B–W equation for an arbitrary half-integral spin $n+1/2$ reads[7]

$$(\partial\!\!\!/ + m)_{\alpha_1\alpha'_1}\, \Psi_{\alpha'_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \qquad (1a)$$

$$(\partial\!\!\!/ + m)_{\beta_1\beta'_1}\, \Psi_{\alpha_1\beta'_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \qquad (1b)$$

$$(\partial\!\!\!/ + m)_{\alpha_2\alpha'_2}\, \Psi_{\alpha_1\beta_1\alpha'_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = 0, \qquad (1c)$$

$$(\partial\!\!\!/ + m)_{\beta_2\beta'_2}\, \Psi_{\alpha_1\beta_1\alpha_2\beta'_2\cdots\alpha_n\beta_n\rho}(x) = 0, \qquad (1d)$$

$$\cdots\cdots$$

$$(\partial\!\!\!/ + m)_{\alpha_n\alpha'_n}\, \Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha'_n\beta_n\rho}(x) = 0, \qquad (1e)$$

$$(\partial\!\!\!/ + m)_{\beta_n\beta'_n}\, \Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta'_n\rho}(x) = 0, \qquad (1f)$$

$$(\partial\!\!\!/ + m)_{\rho\rho'}\, \Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho'}(x) = 0, \qquad (1g)$$

where $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is a completely symmetric multispinor of rank $2n+1$. In order to solve this set of equations in coordinate representation, we first transform them into a form more easily solved by generalizing the procedure used to deal with the B–W equation for spin 5/2.[9]

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On the one hand, by utilizing Dirac equations (1a) and (1b), (1c) and (1d),..., (1e) and (1f), and taking into account the symmetry requirement of $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ in the indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$, $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is expanded as

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = \prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}\,\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x), \quad (2)$$

where $C = \gamma_2\gamma_4$ is the charge conjugation matrix, $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$ are the symmetric matrices, and $\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x)$ is the tensor-spinor satisfying the following equations

$$(\Box - m^2)\,\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (3a)$$

$$\partial_{\nu_i}\,\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x) = 0 \quad (i = 1, 2, \cdots, n), \quad (3b)$$

$$\Psi_{\rho}^{\cdots\nu\nu\cdots}(x) = 0, \quad (3c)$$

$$\Psi_{\rho}^{\cdots\nu_i\nu_{i+1}\cdots}(x) = \Psi_{\rho}^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i = 1, 2, \cdots, n-1). \quad (3d)$$

On the other hand, $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ satisfies the Dirac equation (1g) concerning the index ρ. Substituting Eq.(2) into Eq.(1g), we have

$$\prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j} \times (\not\partial + m)_{\rho\rho'}\,\Psi_{\rho'}^{\nu_1\nu_2\cdots\nu_n}(x) = 0.$$

Because of the independence of matrices $\gamma_\nu C$ and $\Sigma_{\mu\nu}C$, this equation leads to (omitting the spinor index ρ)

$$(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0. \quad (3e)$$

Furthermore, the right-hand side of Eq.(2) is symmetric with indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n$. In order to make sure that $\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x)$ is completely symmetric with all the $2n+1$ indices $\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho$, we further require that the right-hand side of Eq.(2) is also symmetric with indices $\alpha_n\beta_n\rho$. A condition for this requirement is that the contraction of the part of $(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n}\,\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x)$ in equation (2) with the three independent antisymmetric Dirac matrices C^{-1}, $C^{-1}\gamma_5$ and $C^{-1}\gamma_5\gamma_\lambda$ with respect to the indices β_n and ρ vanish, namely

$$(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times \Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1})_{\beta_n\rho} = 0, \quad (4a)$$

$$(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times \Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1}\gamma_5)_{\beta_n\rho} = 0, \quad (4b)$$

$$(\mathrm{i}m\gamma_{\nu_n}C + \Sigma_{\mu_n\nu_n}C\partial_{\mu_n})_{\alpha_n\beta_n} \times \Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x)(C^{-1}\gamma_5\gamma_\lambda)_{\beta_n\rho} = 0. \quad (4c)$$

Expanding Eqs.(4) and using Eqs.(3a)–(3d), we find (omitting the spinor index ρ)

$$\gamma_{\nu_n}(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (5a)$$

$$-2m\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x) + \gamma_{\nu_n}(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (5b)$$

$$(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0. \quad (5c)$$

Both Eqs.(5a) and (5c) are equivalent to Eq.(3e), and the last term of Eq.(5b) vanishes while the first term gives

$$\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x) = 0. \quad (3f)$$

Combining all the above results, we obtain the wave equations for spin $n + 1/2$

$$\Psi_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_n\beta_n\rho}(x) = \prod_{j=1}^{n}(\mathrm{i}m\gamma_{\nu_j}C + \Sigma_{\mu_j\nu_j}C\partial_{\mu_j})_{\alpha_j\beta_j}\,\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x), \quad (6)$$

where $\Psi_{\rho}^{\nu_1\nu_2\cdots\nu_n}(x)$ is a rank n tensor-spinor satisfying the following equations (the spinor index ρ is suppressed)

$$(\Box - m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (7a)$$

$$\partial_{\nu_i}\,\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x) = 0 \quad (i = 1, 2, \cdots, n), \quad (7b)$$

$$\Psi^{\cdots\nu\cdots\nu\cdots}(x) = 0, \quad (7c)$$

$$\Psi^{\cdots\nu_i\nu_{i+1}\cdots}(x) = \Psi^{\cdots\nu_{i+1}\nu_i\cdots}(x) \quad (i = 1, 2, \cdots, n-1), \quad (7d)$$

$$(\not\partial + m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = 0, \quad (7e)$$

$$\gamma_\nu\,\Psi^{\nu\nu_2\nu_3\cdots\nu_n}(x) = 0. \quad (7f)$$

3. The Lagrangian density

The Lagrangian density for fields with arbitrary half-integral spin could be expressed as

$$\begin{aligned} L(x) = &-\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)(\not\partial + m)\,\Psi^{\mu\nu_2\cdots\nu_n}(x) \\ &+ \frac{1}{3}\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\,\Psi^{\nu\nu_2\cdots\nu_n}(x) \\ &- \frac{1}{3}\overline{\Psi}^{\mu\nu_2\cdots\nu_n}(x)\gamma_\mu(\not\partial - m)\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x), \end{aligned} \quad (8)$$

with $\overline{\Psi}^{\nu_1\nu_2\cdots\nu_n}(x) = g_{\nu_1\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_n\mu_n}\gamma_2(\Psi^{\mu_1\mu_2\cdots\mu_n}(x))^{+}$, $\Psi^{\nu_1\nu_2\cdots\nu_i\cdots\nu_n}(x)$ a symmetric tensor-spinor satisfying the condition $\Psi^{\cdots\nu\cdots\nu\cdots}(x) = 0$. Substituting Eq.(8) into the Euler–Lagrange equations

$$\partial_\mu\frac{\partial L}{\partial(\partial_\mu\overline{\Psi}^{\nu\nu_2\cdots\nu_n})} = \frac{\partial L}{\partial\overline{\Psi}^{\nu\nu_2\cdots\nu_n}},$$

we have

$$-(\partial\!\!\!/+m)\,\Psi^{\mu\nu_2\cdots\nu_n}(x)+\frac{1}{3}\gamma_\mu A +\frac{1}{3}\partial_\mu B-\frac{1}{3}\gamma_\mu(\partial\!\!\!/-m)B=0, \tag{9}$$

where

$$A=\partial_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x),\ B=\gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x). \tag{10}$$

Multiplying Eq.(9) by γ_μ gives

$$2A=mB. \tag{11}$$

With ∂_μ acting on Eq.(9) and with the aid of Eq.(11), we have

$$A=0. \tag{12}$$

From Eqs.(9)–(12), the following field equations could be reproduced

$$(\Box-m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0,\quad \partial_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x)=0,$$

$$(\partial\!\!\!/+m)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=0,\quad \gamma_\nu\,\Psi^{\nu\nu_2\cdots\nu_n}(x)=0,$$

which are in agreement with Eqs.(7).

4. Solution to the B–W equation for spin $n+1/2$

We now generalize the method[9] used to solve the B–W equation for spin 5/2 to the more general cases for spin $n+1/2$. We begin with $\Psi^{\nu_1\nu_2\cdots\nu_n}(x)$ which is expanded into plane waves

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{\mathrm{i}px}\,\Psi^{\nu_1\nu_2\cdots\nu_n}(p). \tag{13}$$

Substituting Eq.(13) into Eq.(7a) yields

$$(p^2+m^2)\,\Psi^{\nu_1\nu_2\cdots\nu_n}(p)=0. \tag{14}$$

By virtue of $x\delta(x)=0$, $\Psi^{\nu_1\nu_2\cdots\nu_n}(p)$ in Eq.(14) could be written as

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(p)\\ =&\delta(p^2+m^2)B^{\nu_1\nu_2\cdots\nu_n}(p)\\ =&\frac{1}{2E}[\delta(E-p_0)+\delta(E+p_0)]B^{\nu_1\nu_2\cdots\nu_n}(p),\end{aligned} \tag{15}$$

where $E=\sqrt{\boldsymbol{p}^2+m^2}$. Inserting Eq.(15) into Eq.(13) and integrating over p_0 gives

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(x)\\ =&\int\frac{\mathrm{d}^3p}{(2\pi)^4}\frac{1}{2E}[\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{r}-\mathrm{i}Et}B^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p},E)\\ &+\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{r}+\mathrm{i}Et}B^{\nu_1\nu_2\cdots\nu_n}(-\boldsymbol{p},-E)],\end{aligned} \tag{16}$$

or in a discrete form with a simplified notation

$$\begin{aligned}&\Psi^{\nu_1\nu_2\cdots\nu_n}(x)\\ =&\frac{1}{\sqrt{V}}\sum_{\vec{p}}[a^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{\mathrm{i}px}+b^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})\mathrm{e}^{-\mathrm{i}px})],\end{aligned} \tag{17}$$

where $a^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ and $b^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})$ correspond to the positive and negative solutions respectively, and V is the normalized volume. Substituting Eq.(17) into Eqs.(7b)–(7d), we obtain equations in momentum representation

$$p_{\nu_i}a^{\nu_1\nu_2\nu_i\cdots\nu_n}(\boldsymbol{p})=0,$$

$$p_{\nu_i}b^{\nu_1\nu_2\nu_i\cdots\nu_n}(\boldsymbol{p})=0\quad(i=1,2,\cdots,n), \tag{18a}$$

$$a^{\cdots\nu\nu\cdots}(\boldsymbol{p})=0,\quad b^{\cdots\nu\nu\cdots}(\boldsymbol{p})=0, \tag{18b}$$

$$a^{\cdots\nu_i\nu_{i+1}\cdots}(\boldsymbol{p})=a^{\cdots\nu_{i+1}\nu_i\cdots}(\boldsymbol{p}),$$

$$b^{\cdots\nu_i\nu_{i+1}\cdots}(\boldsymbol{p})=b^{\cdots\nu_{i+1}\nu_i\cdots}(\boldsymbol{p}). \tag{18c}$$

Utilizing $p_\nu e^\nu_\lambda(\boldsymbol{p})=0$, the solution to Eq.(18a) could be expressed as

$$a^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=e^{\nu_1}_{\lambda_1}(\boldsymbol{p})e^{\nu_2}_{\lambda_2}(\boldsymbol{p})\cdots e^{\nu_n}_{\lambda_n}(\boldsymbol{p})a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$$

$$(\lambda_i=1,0,-1), \tag{19a}$$

$$b^{\nu_1\nu_2\cdots\nu_n}(\boldsymbol{p})=\overline{e}^{\nu_1}_{\lambda_1}(\boldsymbol{p})\overline{e}^{\nu_2}_{\lambda_2}(\boldsymbol{p})\cdots\overline{e}^{\nu_n}_{\lambda_n}(\boldsymbol{p})b^+_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p}), \tag{19b}$$

where $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ need to be further determined. $e^\nu_\lambda(\boldsymbol{p})$ are the eigenstates of the helicity operator $\boldsymbol{S}\cdot\dfrac{\hat{\boldsymbol{p}}}{|\boldsymbol{p}|}$ with eigenvalues $\lambda=1,0,-1$, and their explicit form has been given previously,[9] while

$$\overline{e}^\nu_\lambda(\boldsymbol{p})=g_{\nu\mu}(e^\nu_\lambda(\boldsymbol{p}))^*,\quad g_{\nu\mu}=\mathrm{diag}\{1,1,1,-1\}. \tag{20}$$

We use Eqs.(18b) and (18c) to determine $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ in Eq.(19) in a step-by-step way. For the simple case of $n=2$, the method used to determine $a_{\lambda_1\lambda_2}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2}(\boldsymbol{p})$ has been presented in Ref.[9], in what follows we further give the procedure that determines $a_{\lambda_1\lambda_2\lambda_3}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2\lambda_3}(\boldsymbol{p})$ for the case of $n=3$ and we then give directly, as a generalization, the result of $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$. For the case of $n=3$, Eqs.(18b) and (18c) could be expressed explicitly as

$$\begin{cases}a^{\nu\nu\nu_3}(\boldsymbol{p})=0,\quad b^{\nu\nu\nu_3}(\boldsymbol{p})=0,\\ a^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=a^{\nu_2\nu_1\nu_3}(\boldsymbol{p}),\quad b^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=b^{\nu_2\nu_1\nu_3}(\boldsymbol{p}),\end{cases} \tag{21a}$$

$$a^{\nu_1\nu\nu}(\boldsymbol{p})=0,\quad b^{\nu_1\nu\nu}(\boldsymbol{p})=0, \tag{21b}$$

$$a^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=a^{\nu_1\nu_3\nu_2}(\boldsymbol{p}),\quad b^{\nu_1\nu_2\nu_3}(\boldsymbol{p})=b^{\nu_1\nu_3\nu_2}(\boldsymbol{p}). \tag{21c}$$

Using the result for the case of $n = 2$ (see Ref.[9]), we find from Eq.(21a)

$$a^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = e^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p})e^{\nu_3}_{\lambda_3}(\boldsymbol{p})a_{\lambda_{12}\lambda_3}(\boldsymbol{p}),$$
$$b^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \overline{e}^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p})\overline{e}^{\nu_3}_{\lambda_3}(\boldsymbol{p})b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}), \qquad (22)$$

where

$$e^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2} e^{\nu_1}_{\lambda_1}(\boldsymbol{p})e^{\nu_2}_{\lambda_2}(\boldsymbol{p})\langle 1,\lambda_1;1,\lambda_2|1,1,2,\lambda_{12}\rangle$$
$$(\lambda_{12} = 2,1,0,-1,-2), \qquad (23a)$$

$$\overline{e}^{\nu_1\nu_2}_{\lambda_{12}}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2} \overline{e}^{\nu_1}_{\lambda_1}(\boldsymbol{p})\overline{e}^{\nu_2}_{\lambda_2}(\boldsymbol{p})\langle 1,\lambda_1;1,\lambda_2|1,1,2,\lambda_{12}\rangle. \qquad (23b)$$

Substituting Eq.(22) into Eq.(21b) yields

$$e^{\nu_1\nu}_{\lambda_{12}}(\boldsymbol{p})e^{\nu}_{\lambda_3}(\boldsymbol{p})a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0,$$
$$e^{\nu_1\nu}_{\lambda_{12}}(\boldsymbol{p})e^{\nu}_{\lambda_3}(\boldsymbol{p})b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0. \qquad (24)$$

By virtue of (see Ref.(9))

$$e^{\nu}_{\lambda}(\boldsymbol{p}) = L^{\nu\nu'}e^{\nu'}_{\lambda}(0), \quad L^{\nu\nu_1}L^{\nu\nu_2} = \delta_{\nu_1\nu_2},$$

Eq.(24) can be rewritten as

$$e^{\nu_1\nu}_{\lambda_{12}}(0)e^{\nu}_{\lambda_3}(0)a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0, \qquad (25a)$$

$$e^{\nu_1\nu}_{\lambda_{12}}(0)e^{\nu}_{\lambda_3}(0)b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = 0. \qquad (25b)$$

Now we focus our attention on the solution to Eq.(25a), and the solution to Eq.(25b) will be obtained in the same way. Equation (25a) indicates that $a_{\lambda_{12}\lambda_3}(\boldsymbol{p})$ is related to two magnetic quantum numbers λ_{12} and λ_3 ($\lambda_{12} = 2,1,0,-1,-2, \lambda_3 = 1,0,-1$). Recalling the Clebsch–Gordon coefficients for coupling two spin angular momenta with spin 2 and spin 1 respectively, a general candidate for $a_{\lambda_{12}\lambda_3}(\boldsymbol{p})$ is

$$a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\boldsymbol{p}) + \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle a_{2m'}(\boldsymbol{p}) + \sum_{m''} \langle 2,\lambda_{12};1,\lambda_3|2,1,1,m''\rangle a_{1m''}(\boldsymbol{p})$$
$$(m = 3,2,1,0,-1,-2,-3;$$
$$m' = 2,1,0-1,-2; \quad m'' = 1,0,-1). \qquad (26)$$

Letting

$$e^{\nu_1\nu_2\nu_3}_{3m}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle, \qquad (27a)$$

$$e^{\nu_2\nu_2\nu_3}_{2m'}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle, \qquad (27b)$$

$$e^{\nu_2\nu_2\nu_3}_{1m''}(0) = \sum_{\lambda_{12},\lambda_3} e^{\nu_1\nu_2}_{\lambda_{12}}(0)e^{\nu_3}_{\lambda_3}(0)\langle 2,\lambda_{12};1,\lambda_3|2,1,1,m''\rangle, \qquad (27c)$$

then Eq.(25a) takes the form

$$e^{\nu_1\nu\nu}_{3m}(0)a_{3m}(\boldsymbol{p}) + e^{\nu_1\nu\nu}_{2m'}(0)a_{2m'}(\boldsymbol{p}) + e^{\nu_1\nu\nu}_{1m''}(0)a_{1m''}(\boldsymbol{p}) = 0. \qquad (28)$$

With the help of the explicit form of Eq.(27) and the following sum relations

$$e^{\nu}_1(0)e^{\nu}_1(0) = e^{\nu}_{-1}(0)e^{\nu}_{-1}(0)$$
$$= e^{\nu}_0(0)e^{\nu}_1(0) = e^{\nu}_1(0)e^{\nu}_0(0)$$
$$= e^{\nu}_{-1}(0)e^{\nu}_0(0) = e^{\nu}_0(0)e^{\nu}_{-1}(0) = 0,$$

$$e^{\nu}_1(0)e^{\nu}_{-1}(0) = e^{\nu}_{-1}(0)e^{\nu}_1(0) = -1,$$
$$e^{\nu}_0(0)e^{\nu}_0(0) = 1,$$

we find

$$e^{\nu_1\nu\nu}_{3m}(0) = 0 \quad (m = 3,2,1,0,-1,-2,-3),$$
$$e^{\nu_1\nu\nu}_{2m'}(0) = 0 \quad (m' = 2,1,0,-1,-2), \qquad (29a)$$

$$e^{\nu_1\nu\nu}_{1m''}(0) = -\sqrt{\frac{5}{3}}e^{\nu_1}_{m''}(0) \neq 0 \quad (m'' = 1,0,-1). \qquad (29b)$$

Substituting Eq.(29) into Eq.(28) gives

$$a_{1m''}(\boldsymbol{p}) = 0. \qquad (30)$$

Thus Eq.(26) is simplified to

$$a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle a_{3m}(\boldsymbol{p}) + \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle a_{2m'}(\boldsymbol{p}); \qquad (31a)$$

and, similarly, Eq.(25b) leads to

$$b^{+}_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2,\lambda_{12};1,\lambda_3|2,1,3,m\rangle b^{+}_{3m}(\boldsymbol{p}) + \sum_{m'} \langle 2,\lambda_{12};1,\lambda_3|2,1,2,m'\rangle b^{+}_{2m'}(\boldsymbol{p}). \qquad (31b)$$

On the other hand, the symmetric condition (21c) requires

$$a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = a_{\lambda_3\lambda_{12}}(\boldsymbol{p}), \quad b^+_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = b^+_{\lambda_3\lambda_{12}}(\boldsymbol{p}). \quad (32)$$

However

$$\langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 3, m\rangle = \langle 1, \lambda_3; 2, \lambda_{12} | 1, 2, 3, m\rangle, \quad (33a)$$

$$\langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 2, m'\rangle = -\langle 1, \lambda_3; 2, \lambda_{12} | 1, 2, 2, m'\rangle. \quad (33b)$$

Substituting Eq.(31) into Eq.(32) and with the aid of Eq.(33), we have

$$a_{2m'}(\boldsymbol{p}) = 0, \quad b^+_{2m'}(\boldsymbol{p}) = 0. \quad (34)$$

Therefore Eq.(31) is simplified to

$$a_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 3, m\rangle a_{3m}(\boldsymbol{p}), \quad (35a)$$

$$b^+_{\lambda_{12}\lambda_3}(\boldsymbol{p}) = \sum_m \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 3, m\rangle b^+_{3m}(\boldsymbol{p}), \quad (35b)$$

and Eq.(22) becomes correspondingly

$$a^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = e^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) a_{3m}(\boldsymbol{p}),$$
$$b^{\nu_1\nu_2\nu_3}(\boldsymbol{p}) = \overline{e}^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) b^+_{3m}(\boldsymbol{p}), \quad (36a)$$

with

$$e^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2\lambda_3} e^{\nu_1}_{\lambda_1}(\boldsymbol{p}) e^{\nu_2}_{\lambda_2}(\boldsymbol{p}) e^{\nu_3}_{\lambda_3}(\boldsymbol{p}) \times \langle 1, \lambda_1; 1, \lambda_2 | 1, 1, 2, \lambda_{12}\rangle \times \langle 2, \lambda_{12}; 1, \lambda_3 | 2, 1, 3, m\rangle, \quad (36b)$$

$$\overline{e}^{\nu_1\nu_2\nu_3}_{3m}(\boldsymbol{p}) = g_{\nu_1\mu_1} g_{\nu_2\mu_2} g_{\nu_3\mu_3} (e^{\mu_1\mu_2\mu_3}_{3m}(\boldsymbol{p}))^* \quad (m = 3, 2, 1, 0, -1, -2, -3), \quad (36c),$$

or (using the Wigner formula for the Clebsch–Gordon coefficients and omitting the index 3)

$$e^{\nu_1\nu_2\nu_3}_{m}(\boldsymbol{p}) = \sum_{\lambda_1\lambda_2\lambda_3=-1}^{1} e^{\nu_1}_{\lambda_1}(\boldsymbol{p}) e^{\nu_2}_{\lambda_2}(\boldsymbol{p}) e^{\nu_3}_{\lambda_3}(\boldsymbol{p}) \delta(\lambda_1+\lambda_2+\lambda_3, m) \times \sqrt{\frac{(3+m)!(3-m)!}{90\prod_{i=1}^{3}(1+\lambda_i)!(1-\lambda_i)!}}. \quad (36d)$$

Generalizing the above procedure, $a_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ and $b^+_{\lambda_1\lambda_2\cdots\lambda_n}(\boldsymbol{p})$ in Eq.(19) can be determined in a step-by-step way. When the final result is inserted in Eq.(17) we have

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} [e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) a_M(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} + \overline{e}^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) b^+_M(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px})], \quad (37)$$

with $M = 0, \pm1, \pm2, \cdots, \pm n,$

$$\begin{aligned} & e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) \\ &= \sum_{\lambda_i=-1}^{1} \prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\boldsymbol{p}) \\ &\quad \times \prod_{i=1}^{n-1} \langle i, \lambda_1+\lambda_2+\cdots\lambda_i; \\ &\quad 1, \lambda_{i+1} | i, 1, i+1, \lambda_1+\lambda_2+\cdots\lambda_{i+1}\rangle \\ &= \sum_{\lambda_1\lambda_2\cdots\lambda_n=-1}^{1} \delta(\lambda_1+\lambda_2+\cdots\lambda_n, M) \\ &\quad \times \sqrt{\frac{2^n(n+M)!(n-M)!}{(2n)!\prod_{i=1}^{n}(1+\lambda_i)!(1-\lambda_i)!}} \prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\boldsymbol{p}). \end{aligned} \quad (38)$$

Substituting Eq.(37) into Eq.(7e) yields

$$(\mathrm{i}\not{p} + m) a_M(\boldsymbol{p}) = 0 \quad (-\mathrm{i}\not{p} + m) b^+_M(\boldsymbol{p}) = 0. \quad (39)$$

These are the Dirac equations for spin 1/2 and their solutions are (see Ref.[9])

$$a_M(\boldsymbol{p}) = u_r(\boldsymbol{p}) a_{M,r}(\boldsymbol{p}),$$
$$b^+_M(\boldsymbol{p}) = \nu_r(\boldsymbol{p}) b^+_{M,r}(\boldsymbol{p}) \quad \left(r = \frac{1}{2}, -\frac{1}{2}\right). \quad (40)$$

Inserting Eq.(40) into Eq.(37) gives

$$\Psi^{\nu_1\nu_2\cdots\nu_n}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{p}} [e^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) u_r(\boldsymbol{p}) a_{M,r}(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} + \overline{e}^{\nu_1\nu_2\cdots\nu_n}_M(\boldsymbol{p}) \nu_r(\boldsymbol{p}) b^+_{M,r}(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px})]. \quad (41)$$

With this expression, Eq.(7f) becomes

$$\gamma_\nu e^{\nu\nu_2\cdots\nu_n}_M(\boldsymbol{p}) u_r(\boldsymbol{p}) a_{M,r}(\boldsymbol{p}) = 0, \quad (42a)$$

$$\gamma_\nu \overline{e}^{\nu\nu_2\cdots\nu_n}_M(\boldsymbol{p}) \nu_r(\boldsymbol{p}) b^+_{M,r}(\boldsymbol{p}) = 0. \quad (42b)$$

Hence Eq.(42a) can be rewritten as (see Ref.[9])

$$\gamma_\nu e^{\nu\nu_2\cdots\nu_n}_M(0) u_r(0) a_{M,r}(\boldsymbol{p}) = 0, \quad (43)$$

where $a_{M,r}(\boldsymbol{p})$ is related to two magnetic quantum numbers M and r $\left(M = 0, \pm1, \pm2, \cdots, \pm n; r = \frac{1}{2}, -\frac{1}{2}\right)$. Recalling the Clebsch–Gordon coefficients for coupling two spin angular momenta with spin n and

spin 1/2 respectively, a general candidate for $a_{M,r}(\boldsymbol{p})$ is

$$\begin{aligned} & a_{M,r}(\boldsymbol{p}) \\ =& \sum_{m} \left\langle n, M; \frac{1}{2}, r | n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle a_{n+\frac{1}{2},m}(\boldsymbol{p}) \\ & + \sum_{m'} \left\langle n, M; \frac{1}{2}, r | n, \frac{1}{2}, n-\frac{1}{2}, m' \right\rangle a_{n-\frac{1}{2},m'}(\boldsymbol{p}) \end{aligned}$$

$$\left(m = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n+\frac{1}{2}\right);\right.$$

$$\left. m' = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n-\frac{1}{2}\right)\right). \tag{44}$$

Letting

$$\begin{aligned} U^{\nu_1\nu_2\cdots\nu_n}_{n+\frac{1}{2},m}(\boldsymbol{p}) =& \sum_{M,r} e^{\nu_1\nu_2\cdots\nu_n}_{M}(\boldsymbol{p}) u_r(\boldsymbol{p}) \\ & \times \left\langle n, M; \frac{1}{2}, r | n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle, \end{aligned} \tag{45a}$$

$$\begin{aligned} U^{\nu_1\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(\boldsymbol{p}) =& \sum_{M,r} e^{\nu_1\nu_2\cdots\nu_n}_{M}(\boldsymbol{p}) u_r(\boldsymbol{p}) \\ & \times \left\langle n, M; \frac{1}{2}, r | n, \frac{1}{2}, n-\frac{1}{2}, m' \right\rangle, \end{aligned} \tag{45b}$$

then Eq.(43) takes the form

$$\begin{aligned} & \gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n+\frac{1}{2},m}(0) a_{n+\frac{1}{2},m}(\boldsymbol{p}) \\ & + \gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(0) a_{n-\frac{1}{2},m}(\boldsymbol{p}) = 0. \end{aligned} \tag{46}$$

However

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n+\frac{1}{2},m}(0) = 0 \quad \left(m = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n+\frac{1}{2}\right)\right), \tag{47a}$$

$$\gamma_\nu U^{\nu\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(0) = \mathrm{i}\gamma_5 \sqrt{\frac{2n+1}{n}} U^{\nu_2\cdots\nu_n}_{n-\frac{1}{2},m'}(0) \neq 0$$

$$\left(m' = \pm\frac{1}{2}, \pm\frac{3}{2}, \cdots, \pm\left(n-\frac{1}{2}\right)\right). \tag{47b}$$

Substituting Eq.(47) into Eq.(46) yields

$$a_{n-\frac{1}{2},m'}(\boldsymbol{p}) = 0, \tag{48}$$

and Eq.(44) becomes $\left(\text{omit the index } n+\frac{1}{2} \text{ in } a_{n+\frac{1}{2},m}(\boldsymbol{p})\right)$

$$a_{M,r}(\boldsymbol{p}) = \sum_{m} \left\langle n, M; \frac{1}{2}, r | n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle a_m(\boldsymbol{p}). \tag{49a}$$

Similarly, Eq.(42b) leads to

$$b^+_{M,r}(\boldsymbol{p}) = \sum_{m} \left\langle n, M; \frac{1}{2}, r | n, \frac{1}{2}, n+\frac{1}{2}, m \right\rangle b^+_m(\boldsymbol{p}). \tag{49b}$$

Inserting Eq.(49) into Eq.(41) gives the final result

$$\begin{aligned} \Psi^{\nu_1\nu_2\cdots\nu_n}(x) =& \frac{1}{\sqrt{V}} \sum_{\vec{p}} [a_m(\boldsymbol{p}) U^{\nu_1\nu_2\cdots\nu_n}_m(\boldsymbol{p}) \mathrm{e}^{\mathrm{i}px} \\ & + b^+_m(\boldsymbol{p}) V^{\nu_1\nu_2\cdots\nu_n}_m(\boldsymbol{p}) \mathrm{e}^{-\mathrm{i}px})], \end{aligned} \tag{50}$$

with

$$\begin{aligned} U^{\nu_1\nu_2\cdots\nu_n}_m =& \sum_{\lambda_1,\lambda_2,\cdots\lambda_n=-1}^{1} \sum_{r=-1/2}^{1/2} \prod_{i=1}^{n} e^{\nu_i}_{\lambda_i}(\boldsymbol{p}) u_r(\boldsymbol{p}) \delta(\lambda_1+\lambda_2+\cdots\lambda_n+r, m) \\ & \times \sqrt{\frac{2^n \left(n+\frac{1}{2}+m\right)! \left(n+\frac{1}{2}-m\right)!}{(2n+1)! \prod_{i=1}^{n} (1+\lambda_i)!(1-\lambda_i)! \left(\frac{1}{2}+r\right)! \left(\frac{1}{2}-r\right)!}}, \end{aligned} \tag{51}$$

$$\begin{aligned} V^{\nu_1\nu_2\cdots\nu_n}_m =& \sum_{\lambda_1,\lambda_2,\cdots\lambda_n=-1}^{1} \sum_{r=-1/2}^{1/2} \prod_{i=1}^{n} \overline{e}^{\nu_i}_{\lambda_i}(\boldsymbol{p}) \nu_r(\boldsymbol{p}) \delta(\lambda_1+\lambda_2+\cdots\lambda_n+r, m) \\ & \times \sqrt{\frac{2^n \left(n+\frac{1}{2}+m\right)! \left(n+\frac{1}{2}-m\right)!}{(2n+1)! \prod_{i=1}^{n} (1+\lambda_i)!(1-\lambda_i)! \left(\frac{1}{2}+r\right)! \left(\frac{1}{2}-r\right)!}}, \end{aligned} \tag{52}$$

where the Wigner formula for the Clebsch–Gordon coefficients has been used in the last step. By utilizing the normalization and orthogonality properties of $e^{\nu}_{\lambda}(\boldsymbol{p})$, $u_r(\boldsymbol{p})$, $\nu_r(\boldsymbol{p})$ and the properties of the Clebsch–Gordon coefficients, it can be shown that both the positive and negative solutions possess normalization and orthogonality properties.

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