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Multi-mode q-oscillator algebras with $q^{2(k+1)}=1$ and related thermo field dynamics*

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A type of multi-mode q-oscillator algebra with $q^{2(k+1)}=1$ is set up and the associated q_k-thermo field dynamics is constructed for all $k=1,2,\cdots,\infty$ in a unified form. It is demonstrated that these q_k-thermo field dynamics can all be nicely fitted into the algebraic formulation of statistical mechanics (axiomatized form for statistical physics). This means that we obtain infinitely many realizations of the algebraic scheme, which extend the consideration of Ojima [1981 *Ann. Phys.* **137** 1] and contain the usual thermo field dynamics for the fermionic ($k=1$) and bosonic ($k=\infty$) systems as special cases. As simple applications, the q_k-statistical average of some operators are given.

Keywords: multi-mode q_k-oscillator algebra, q_k-thermo field dynamics, algebraic formulation of statistical mechanics

PACC: 0590, 0290, 0530

1. Introduction

Statistical properties, e.g. the Bose–Einstein (BE) and/or Fermi–Dirac (FD) statistics, play a very important role in quantum, as well as statistical, mechanics. Recently, much attention has been paid, from different points of view, to some generalized statistics (see, for example, Refs.[1]–[4]). For these studies, the theories of quantum groups and the related noncommutative algebras[5−9] provide a kind of powerful mathematical tool. On the other hand, the thermo field dynamics (TFD),[10−13] as an operator formulation of the finite temperature field theory,[13−16] is very effective and fruitful for studying the usual bosonic or/and fermionic systems. As shown by Ojima,[12] the usual TFD are some realizations of the algebraic formulation of statistical mechanics (AFSM)[17,18] (axiomatized form for statistical physics); however, the concrete formulae for bosonic and fermionic systems are different.

The statistical mechanical properties of some q-deformation system and the connection between some q-algebras and structures of bosonic TFD were considered, respectively, in Refs.[19] and [20], where q was taken to be real. In a previous paper,[21] we gave a sketchy construction and discussion of the so-called q_k-TFD for q_k-particle systems described by the single-mode q-oscillator algebras[22−24]

$$aa^{+}-q^{\pm 1}a^{+}a=q^{\mp N},$$

$$[N,\ a]=-a, \qquad [N,\ a^{+}]=a^{+}, \tag{1}$$

where q is a root of unity such that

$$q^{2(k+1)}=1 \quad (\text{i.e. } q=\mathrm{e}^{i\pi/(k+1)})$$

$$k\in\{1,2,\cdots,\infty\}. \tag{2}$$

However, to obtain a relevant field theory construction, it is necessary to generalize these discussions to a multi-mode form, and this is the main aim of the present paper. Unlike the single-mode case, there is no ready-made multi-mode q-algebra with $q^{2(k+1)}=1$ suitable for our studies. So we first construct such a multi-mode q-algebra and then construct the related TFD.

In the preliminaries below, we recall briefly the related results of AFSM and the usual TFD; the details can be found in, for example, Refs.[10]–[13], [17] and [18]. In section 2, to obtain a multi-mode generalization of the q_k-oscillator algebra (1), we give the differential calculus on the m-dimensional quantum hyperplane with $q^{2(k+1)}=1$. Section 3 sets up the desired multi-mode q_k-oscillator algebras and their Fock

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representations. In section 4, the related q_k-TFD for all $k = 1, 2, \cdots, \infty$ are constructed in a unified way, and then we demonstrate that all of these q_k-TFD can be fitted nicely into the AFSM. This gives a series of extensions of the discussion in Ref.[12]. Section 5 presents some simple applications. Section 6 is devoted to conclusions and discussion.

Preliminaries

1.1. Related algebraic theory[17]

Let $\mathcal{U}$ be a C^*-algebra describing some physical system under consideration; it can be represented as an operator algebra acting on a suitable Hilbert space $\mathcal{H}$ by, for example, the Gelfand–Naimark–Segal (GNS) construction. To save notation, in this paper we do not stick to the distinction between an operator algebra itself and its representation. Let $\mathcal{M}$ denote the enveloping von Neumann algebra $\mathcal{U}''$ (the double commutant of $\mathcal{U}$), in general $\mathcal{M} \supset \mathcal{U}$. To our purpose, $\mathcal{M}$ is assumed to be σ-finite.[13,17] According to the standard theorems,[17] such an algebra $\mathcal{M}$ admits a cyclic and separating vector Ω such that

$$\mathcal{H} = \overline{\mathcal{U}\Omega} = \overline{\mathcal{M}\Omega} = \overline{\mathcal{M}'\Omega}, \tag{3}$$

$$\pi(A)\Omega = 0 \Longrightarrow A = 0 \text{ for } A \in \mathcal{U}, \tag{4}$$

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. From Eq.(3) one can define two antilinear operators S and F satisfying[12,13,17]

$$\begin{aligned} &SA\Omega = A^+\Omega, \qquad FA'\Omega = A'^+\Omega, \\ &A \in \mathcal{M}, \quad A' \in \mathcal{M}', \\ &S^+ = F = F^{-1}, \quad F^+ = S = S^{-1}. \end{aligned} \tag{5}$$

The polar decompositions

$$S = J\Delta^{1/2}, \qquad \Delta = S^+S = FS \tag{6}$$

define the modular operator Δ and the modular conjugation operator J. They are characterized uniquely by the following conditions[12,13,17]

$$\begin{aligned} &(J1)\ (J\Phi, J\Psi) = (\Phi, \Psi)^*, && (\Delta 1)\Delta^+ = \Delta, \\ &(J2)\ J^2 = 1, && (\Delta 2)\ \Delta > 0, \\ &(J3)\ J\mathcal{M}J = \mathcal{M}', && (\Delta 3)\ \Delta^{\mathrm{i}t}\mathcal{M}\Delta^{-\mathrm{i}t} = \mathcal{M}, \\ &(J4)\ J\Omega = \Omega, && (\Delta 4)\ \Delta\Omega = \Omega, \\ &(J5)\ (QJQJ\Omega, \Omega) \geq 0 \\ &(\Delta 5)\ (\Omega,\ \sigma_t[B]A\Omega) = (\Omega,\ A\sigma_{t-i}[B]\Omega), \end{aligned} \tag{7}$$

where $\Phi,\ \Psi \in \mathcal{H}$, $(\ ,\ )$ denotes the inner product on $\mathcal{H}$ and

$$\sigma_t[A] := \Delta^{\mathrm{i}t}A\Delta^{-\mathrm{i}t}. \tag{8}$$

Moreover, from Eqs.(5) and (6) we have

$$J\Delta J = \Delta^{-1}. \tag{9}$$

When using the above theory to statistical physics, it is usually taken that[12,18]

$$\begin{aligned} &\langle A\rangle = (\Omega,\ A\Omega), \qquad \Delta = \mathrm{e}^{-\beta\overline{H}}, \\ &\overline{H} = H - JHJ, \end{aligned} \tag{10}$$

where $\langle A\rangle$ is the thermal average of operator A, H is Hamiltonian and β is the inversed temperature.

1.2. Basic theories of TFD[10−13,25]

The basic idea of TFD is to introduce a thermal vacuum state $|0(\beta)\rangle$ such that the thermal average of an operator $A \in \mathcal{U}$ can be expressed as an expectation value with respect to $|0(\beta)\rangle$

$$\langle A\rangle = \langle 0(\beta)\,|A|\,0(\beta)\rangle\,. \tag{11}$$

In order to have a consistent operator formalism, the doubling of the operator degrees of freedom is needed; corresponding to an operator A, a tilde operator $\tilde{A}$ is introduced. Having the two sets of $\mathcal{U} = \{A\}$ and $\tilde{\mathcal{U}} = \{\tilde{A}\}$, one can construct TFD with the following axioms.[25,26]

(i) The tilde conjugation rules

$$\begin{aligned} &(AB)^\sim = \tilde{A}\tilde{B}, \quad (c_1A + c_2B)^\sim = c_1^*\tilde{A} + c_2^*\tilde{B}, \\ &(A^+)^\sim = \tilde{A}^+, \end{aligned} \tag{12}$$

where c_1, c_2 are complex numbers.

(ii)
$$[A,\ \tilde{B}]_\mp \equiv A\tilde{B} \mp \tilde{B}A = 0, \tag{13}$$

where $-(+)$ for bosonic (fermionic) operators.

(iii)
$$(|0(\beta)\rangle)^\sim = |0(\beta)\rangle\,. \tag{14}$$

(iv) The time translation of $Q \in \mathcal{U}$ or $\tilde{\mathcal{U}}$ is given by

$$A(t) = \mathrm{e}^{\mathrm{i}\overline{H}t}A\mathrm{e}^{-\mathrm{i}\overline{H}t}, \quad \overline{H} = H - \tilde{H}, \tag{15}$$

where H is the Hamiltonian.

(v) The thermal state condition

$$\begin{aligned} &A(t)\,|0(\beta)\rangle = \alpha\tilde{A}^+(t - \mathrm{i}\beta/2)\,|0(\beta)\rangle\,, \\ &\langle 0(\beta)|\,A(t) = \langle 0(\beta)|\,\tilde{A}^+(t + \mathrm{i}\beta/2)\alpha^*. \end{aligned} \tag{16}$$

(vi) The double tilde conjugation is

$$\tilde{\tilde{A}} = A. \tag{17}$$

There is a certain freedom for the choice of α in Eq.(16) and the double tilde conjugation in Eq.(17),[26] which we use in the following.

2. The q-differential calculus on a quantum hyperplane with $q^{2(k+1)}=1$

According to Ref.[27], if we are given matrices $B,\ C,\ F \in \mathrm{End}(C^m \otimes C^m)$ fulfilling the following consistency conditions

$$\begin{aligned}
&B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23},\\
&C_{12}C_{23}C_{12} = C_{23}C_{12}C_{23},\\
&B_{12}C_{23}C_{12} = C_{23}C_{12}B_{23},\\
&(B_{12}-I)(C_{12}+I) = 0; \qquad (18a)\\
&F_{12}F_{23}F_{12} = F_{23}F_{12}F_{23},\\
&C_{12}C_{23}F_{12} = F_{23}C_{12}C_{23},\\
&(C_{12}+I)(F_{12}-I) = 0; \qquad (18b)
\end{aligned}$$

where $B_{12} = B\otimes I,\ B_{23} = I\otimes B$, etc, then we can construct a consistent differential calculus on an algebra generated by $\{x^i,\ i=1,\cdots,m\}$ such that

$$\begin{aligned}
&x_1x_2 = B_{12}x_1x_2, \quad x_1\mathrm{d}x_2 = C_{12}(\mathrm{d}x_1)x_2,\\
&\mathrm{d}x_1\mathrm{d}x_2 = -C_{12}\mathrm{d}x_1\mathrm{d}x_2,\\
&D_2D_1 = D_2D_1F_{12}, \quad D_jx^i = \delta^i_j + C^{ik}_{jl}x^lD_k,\\
&D_j\mathrm{d}x^i = (C^{-1})^{ik}_{jl}\mathrm{d}x^lD_k, \qquad (19)
\end{aligned}$$

where the standard tensor notations are used. In formulae (19) $\mathrm{d}x^i,\ D_i,\ i=1,\cdots,m$ are referred to as q-(exterior) differentials and q-derivatives, respectively.

The above $x^i,\ \mathrm{d}x^i,\ D_i,\quad i=1,\cdots,m$ are generally regarded as algebraic elements; for our present purpose, we need only to consider the subalgebra generated by $\{x^i,\ D_i,\ i=1,\cdots,m\}$. Now our task is to look for a set of matrices $B,\ C,\ F$ satisfying Eqs.(18) and suitable to our construction of multi-mode q_k-oscillator algebras. Motivated by Ref.[28], we take

$$\begin{aligned}
B = F =& \sum_i e^i_i\otimes e^i_i + q^2\sum_{i<j}\mathrm{e}^i_j\otimes e^j_i\\
&+ q^{-2}\sum_{i>j}e^i_j\otimes e^j_i,\\
C(p) =& \sum_i p_ie^i_i\otimes e^i_i + q^2\sum_{i<j}e^i_j\otimes e^j_i\\
&+ q^{-2}\sum_{i>j}e^i_j\otimes e^j_i, \qquad (20)
\end{aligned}$$

where $e^i_j \in \mathrm{Mat}(\mathbf{C}^m)$ are matrix units, $p_i\in\mathbf{C},\ i=1,\cdots,m$ and q is the same as in Eq.(2). The fact that B and $C(p)$ fulfil Eqs.(18a) has been pointed out in Ref.[28]. Here we show that F and $C(p)$ in Eqs.(20) also satisfy Eqs.(18b). Since $F=B$, we need only to verify the second and third equations.

From Eqs.(20), it follows that

$$F^+(=B^+) = F(=B), \quad C^+(p) = C(p^*), \qquad (21)$$

where "+" represents the hermitian conjugation of the matrices. Because $p_i,\ i=1,\cdots,m$ and q in Eqs.(20) are independent of each other, by Eqs.(18a) and (21) we have

$$\begin{aligned}
&B_{12}C_{23}(p)C_{12}(p) = C_{23}(p)C_{12}(p)B_{23}\\
&\Longleftrightarrow B_{12}C_{23}(p^*)C_{12}(p^*) = C_{23}(p^*)C_{12}(p^*)B_{23}\\
&\underset{B=F}{\overset{+}{\Longleftrightarrow}} C_{12}(p)C_{23}(p)F_{12} = F_{23}C_{12}(p)C_{23}(p), \qquad (22a)
\end{aligned}$$

and

$$\begin{aligned}
&(B_{12}-I)(C_{12}(p)+I) = 0\\
&\Longleftrightarrow (B_{12}-I)(C_{12}(p^*)+I) = 0\\
&\underset{B=F}{\overset{+}{\Longleftrightarrow}} (C_{12}(p)+I)(F_{12}-I) = 0. \qquad (22b)
\end{aligned}$$

So far, the parameters p_i are arbitrary. Now we consider the case of $p_i = q^2,\ i=1,\ \cdots,\ m$. Then, by using Eqs.(19) and (20), we can write explicitly the algebraic relations for $x^i,\ D_i,\ i=1,\ \cdots,\ m$ as follows

$$\begin{aligned}
&x^ix^j = q^2x^jx^i, \quad D_iD_j = q^2D_jD_i,\\
&D_jx^i = q^2x^iD_j, \quad D_ix^j = q^{-2}x^jD_i
\end{aligned} \qquad i<j,$$

$$\begin{aligned}
&D_ix^i - q^2x^iD_i = 1 \qquad (\text{no sum on } i),\\
&i,j = 1,\cdots,m. \qquad (23)
\end{aligned}$$

3. Multi-mode q_k-oscillator algebras and their Fock representations

We introduce the raising and lowering operators $\{\overline{b}_i\}$ and $\{b_i\}$ by the correspondence

$$x^i \longrightarrow \overline{b}_i, \quad D_i \longrightarrow b_i \quad i=1,\ \cdots,\ m. \qquad (24)$$

Then from Eqs.(23), their commutation relations

$$\begin{aligned}
&\overline{b}_i\overline{b}_j = q^2\overline{b}_j\overline{b}_i, \quad b_ib_j = q^2b_jb_i,\\
&b_j\overline{b}_i = q^2\overline{b}_ib_j, \quad b_i\overline{b}_j = q^{-2}\overline{b}_jb_i,
\end{aligned} \qquad i<j,$$

$$\begin{aligned}
&b_i\overline{b}_i - q^2\overline{b}_ib_i = 1 \qquad (\text{no sum on } i)\\
&i,\ j = 1,\cdots,m, \qquad (25)
\end{aligned}$$

follow immediately. The discussion in section 2 has shown that Eqs.(25) give an algebra on complex number field C.

For physical applications, it is convenient to introduce the number operators $N_i,\ i=1,\ \cdots,\ m$ such that

$$[N_i,\ \overline{b}_j]=\delta_{ij}\overline{b}_j,\quad [N_i,\ b_j]=-\delta_{ij}b_j,$$
$$[N_i,\ N_j]=0\qquad i,j=1,\ \cdots,\ m. \tag{26}$$

For each i, $\{b_i,\ \overline{b}_i,\ N_i\}$ generates a single-mode oscillator algebra with a complex parameter q^2.[29,24] Since the additional operator N_i (with fixed i) is commutable with all operators that do not involve the operators b_i and $\overline{b}_i$, then $\{b_i,\ \overline{b}_i,\ N_i,\ i=1,\cdots,m\}$ with relations (25) and (26) generate a multi-mode algebra, and each single-mode oscillator algebra mentioned above is its subalgebra.

However, in the above multi-mode algebra, the operators $\overline{b}_i$ and b_i cannot be hermitianly conjugated to each other.[30] To circumvent this problem, we introduce the a-annihilation and creation operators $a_i,\ a_i^+$ as

$$a_i=q^{-N_i/2}b_i,\quad a_i^+=\overline{b}_iq^{-N_i/2}\qquad i=1,\ldots,m \tag{27}$$

and from formulae (25) and (26) we obtain the following relations

$$a_ia_j=q^2a_ja_i,\qquad a_i^+a_j^+=q^2a_j^+a_i^+,$$
$$a_ja_i^+=q^2a_i^+a_j\qquad i<j,$$
$$[N_i,\ a_j]=-\delta_{ij}a_j,$$
$$[N_i,\ a_j^+]=\delta_{ij}a_j^+,\quad [N_i,\ N_j]=0,$$
$$a_ia_i^+-q^{\pm1}a_i^+a_i=q^{\mp N_i}\qquad (\text{no sum on } i)$$
$$i,j=1,\cdots,m. \tag{28}$$

As useful consequences of Eqs.(28) we also have

$$a_i^+a_i=[N_i],\quad a_ia_i^+=[N_i+1]\quad i=1,\ \cdots,\ m, \tag{29}$$

where we have used the abbreviation for an operator or a number X

$$[X]:=\frac{q^X-q^{-X}}{q-q^{-1}}. \tag{30}$$

The Fock representation of multi-mode q-algebra (28) can be constructed as follows. Let $|0\rangle=|0,\cdots,0\rangle$ be the vacuum state defined by

$$N_i|0\rangle=0,\quad a_i|0\rangle=0\qquad i=1,\ \cdots,\ m, \tag{31}$$

and let $\{|n_1,\cdots,n_m\rangle,\ n_1,\cdots,n_m=0,1,\cdots\}$ be the completely orthonormal set of common eigenstates of $\{N_i\}$ such that

$$N_i|n_1,\cdots,n_i,\cdots,n_m\rangle=n_i|n_1,\cdots,n_i,\cdots,n_m\rangle,$$
$$\langle n_1',\cdots,n_i',\cdots,n_m'|n_1,\cdots,n_i,\cdots,n_m\rangle=\prod_{i=1}^m\delta_{n_i'n_i}. \tag{32}$$

Then from Eqs.(28) we can obtain, up to total phase factors,

$$\begin{aligned}&a_j|n_1,\cdots,n_j,\cdots,n_m\rangle\\ =&q^{2\sum_{i<j}n_i}[n_j]^{1/2}|n_1,\cdots,n_j-1,\cdots,n_m\rangle,\\ &a_j^+|n_1,\cdots,n_j,\cdots,n_m\rangle\\ =&q^{-2\sum_{i<j}n_i}[n_j+1]^{1/2}|n_1,\cdots,n_j+1,\cdots,n_m\rangle.\end{aligned} \tag{33}$$

Thus, Eqs.(28) and the fact that $[k+1]=0$ imply that we can consistently set

$$(a_i^+)^{k+1}=0,\quad a_i^{k+1}=0\qquad i=1,\cdots,m. \tag{34}$$

From Eqs.(28), (34) and (31)–(33) we can write

$$\begin{aligned}&|n_1,\cdots,n_i,\cdots,n_m\rangle\\ =&\frac{(a_1^+)^{n_1}\cdots(a_i^+)^{n_i}\cdots(a_m^+)^{n_m}}{([n_1]!\cdots[n_i]!\cdots[n_m]!)^{1/2}}\Big|0\Big\rangle\\ &n_i(1\le i\le m)=0,\ 1,\ \cdots,\ k.\end{aligned} \tag{35}$$

The relations (28) and (34) with Eq.(2) give infinitely many algebras; for each $k\in\{1,2,\cdots,\infty\}$, Eq.(2) determines a value of q and Eqs.(28) and (34) constitute a corresponding m-mode q-algebra. We call it m-mode q_k-oscillator algebras and denote it by $\mathcal{U}_{k;m}$ ($k=1,2,\cdots,\infty;\ m=1,2,\cdots$). It is this $\mathcal{U}_{k;m}$ that we use in the following. From the point of view of particle physics, each $\mathcal{U}_{k;m}$ describes a type of particle (which we call q_k-particles) obeying certain generalized statistics (which we call q_k-statistics). Formulae (34) and (35) show that, in the q_k-particle system, a fixed state can hold at most k identical ith-mode q_k-particles. This is a generalization of the usual Pauli exclusion principle, which we call the q_k-exclusion principle. When $k=1$ and ∞, $\mathcal{U}_{k;m}$ give the usual multi-mode fermionic and bosonic algebras, respectively. However, when k takes another integer, the corresponding q_k-particle system obeys the so-called q_k-statistics different from the ordinary FD and BE ones. In the next section, we construct a type of q_k-TFD suitable to the q_k-particle system in a unified form for all $k=1,2,\cdots,\infty$ and then show that they may all be nicely fitted into the frame of AFSM.

4. q_k-TFD for multi-mode q_k-particle systems

We first point out that the multi-mode q_k-algebras $\mathcal{U}_{k;m}$ is a C^*-algebra under the involution* defined by

$$* : A \longrightarrow A^* := A^+ \in \mathcal{U}_{k;m}, \quad \forall A \in \mathcal{U}_{k;m}, \tag{36}$$

and, according to the standard theories,[17] its enveloping von Neumann algebra $\mathcal{M}_{k;m} := \mathcal{U}''_{k;m}$ is σ-finite. Then from Tomita–Takesaki modular theory (cf Prel. 1.1 in section 1) we can have the corresponding Ω, J, Δ and the commutant $\mathcal{U}'_{k;m}$, etc.

In order to construct the formulation of the q_k-TFD desired, we use a multi-mode q_k-Klein transformation. Defining

$$\mathcal{N} := \sum_{i=1}^{m} N_i, \ \theta := q^{2\mathcal{N}}, \ \tilde{\theta} := J\theta J, \ \overline{\theta} := \theta\tilde{\theta}, \tag{37}$$

we have

$$\theta^+ = \theta^{-1}, \ \ \tilde{\theta}^+ = \tilde{\theta}^{-1}, \ \ \overline{\theta}^+ = \overline{\theta}^{-1}, \tag{38}$$

and

$$\theta a_i^+ \theta^{-1} = q^2 a_i^+, \quad \theta a_i \theta^{-1} = q^{-2} a_i,$$
$$\tilde{\theta} J a_i^+ J \tilde{\theta}^{-1} = q^{-2} J a_i^+ J, \quad \tilde{\theta} J a_i J \tilde{\theta}^{-1} = q^2 J a_i J$$
$$i = 1, \ \cdots, \ m. \tag{39}$$

With these notations we introduce the tilde operators as

$$\tilde{a}_i := qJa_iJ\overline{\theta}, \quad \widetilde{a^+}_i := qJa_i^+J\overline{\theta}^+ = (\tilde{a}_i)^+$$
$$i = 1, \ \cdots, \ m. \tag{40}$$

From Eqs.(37)–(40) and the fact that $[A, \ JBJ] = 0, \ A, B \in \mathcal{M}_{k;m}$, it follows that

$$a_i\tilde{a}_j = q^2\tilde{a}_j a_i, \quad a_i\tilde{a}_j^+ = q^{-2}\tilde{a}_j^+ a_i,$$
$$a_i^+\tilde{a}_j = q^{-2}\tilde{a}_j a_i^+, \quad a_i^+\tilde{a}_j^+ = q^2\tilde{a}_j^+ a_i^+$$
$$i, \ j = 1, \ \cdots, \ m \tag{41}$$

and

$$\tilde{a}_i^n(\tilde{a}_i^+)^n = Ja_i^n(a_i^+)^nJ, \quad (\tilde{a}_i\tilde{a}_i^+)^n = J(a_ia_i^+)^nJ,$$
$$(\tilde{a}_i^+)^n\tilde{a}_i^n = J(a_i^+)^n a_i^n J, \quad (\tilde{a}_i^+\tilde{a}_i)^n = J(a_i^+a_i)^nJ$$
$$i = 1, \cdots, m. \tag{42}$$

Relations (29) and (42) imply that

$$\tilde{N}_i = JN_iJ \qquad i = 1, \cdots, m \tag{43}$$

are the tilde number operators in the q_k-TFD.

From the above relations we have

$$\tilde{a}_i\tilde{a}_j = q^2\tilde{a}_j\tilde{a}_i, \quad \tilde{a}_i^+\tilde{a}_j^+ = q^2\tilde{a}_j^+\tilde{a}_i^+,$$
$$\tilde{a}_j\tilde{a}_i^+ = q^2\tilde{a}_i^+\tilde{a}_j \quad i > j,$$
$$[\tilde{N}_i, \ \tilde{a}_j] = -\delta_{ij}\tilde{a}_j,$$
$$[\tilde{N}_i, \ \tilde{a}_j^+] = \delta_{ij}\tilde{a}_j^+, \quad [\tilde{N}_i, \ \tilde{N}_j] = 0,$$
$$(\tilde{a}_i^+)^{k+1} = 0, \qquad \tilde{a}_i^{k+1} = 0,$$
$$\tilde{a}_i\tilde{a}_i^+ - q^{\pm 1}\tilde{a}_i^+\tilde{a}_i = q^{\mp\tilde{N}_i} \quad \text{(no sum on } i\text{)}$$
$$i, j = 1, \cdots, m. \tag{44}$$

The relations (44) and condition (2) present the tilde conjugation algebra $\widetilde{\mathcal{U}}_{k;m}$ in the q_k-TFD for multi-mode systems. Based on Eqs.(40) and (43), the tilde operator $\tilde{A}$ corresponding to any operator $A \in \mathcal{U}_{k;m}$ can be easily obtained by the rules in (12). And similar to Eqs.(31)–(33) and (35), we can construct the Fock representation of $\widetilde{\mathcal{U}}_{k;m}$.

By exploiting the freedom of α in Eqs.(16) and (17), we can take

$$(\tilde{a}_i)^\sim = qJ\tilde{a}_iJ\overline{\theta}^+, \qquad (\tilde{a}_i^+)^\sim = qJ\tilde{a}_i^+J\overline{\theta}$$
$$i = 1, \cdots, m. \tag{45}$$

Thus we have

$$\tilde{\tilde{a}}_i = a_i, \quad \tilde{\tilde{a}}_i^+ = a_i^+, \quad \tilde{\tilde{N}}_i = N_i \quad i = 1, \cdots, m \tag{46a}$$

and further

$$\tilde{\tilde{A}} = A, \qquad \forall A \in \mathcal{U}_{k;m}. \tag{46b}$$

The relations (46) are consistent with all of the other formulae. Moreover, the tilde operations (40) and (45) can be written in a compact form in terms of the "q_k-thermal doublet"[12,13,25] $a_i := \begin{pmatrix} a_i \\ q\tilde{a}_i^+ \end{pmatrix}$:

$$\tilde{a}_i = qJa_iJ\overline{\theta}, \ \ \widetilde{a_i^+} = qJa_i^+J\overline{\theta}^+ = (\widetilde{a}_i)^+$$
$$i = 1, \ \cdots, \ m. \tag{47}$$

The thermal vacuum state $|0(\beta)_m\rangle_k$ of q_k-TFD for multi-mode q_k-systems can be introduced as (still denoting $|0\rangle \otimes |\tilde{0}\rangle$ by $|0\rangle$)

$$|0(\beta)_m\rangle_k = Z_k^{-1/2}(\beta; m)|\Theta\rangle,$$
$$|\Theta\rangle := \sum_{n_1, \cdots, n_m = 0}^{k} \exp\left[-\frac{1}{2}\beta E(n_1, \cdots, n_m)\right]$$
$$\times \frac{(q^{-1}a_1^+\tilde{a}_1^+)^{n_1} \cdots (q^{-1}a_m^+\tilde{a}_m^+)^{n_m}}{[n_1]! \cdots [n_m]!}\Big|0\Big\rangle,$$

$$Z_k(\beta;m) = \langle\Theta|\Theta\rangle = \sum_{n_1,\cdots,n_m=0}^{k} \exp(-\beta E(n_1,\cdots,n_m)). \quad (48)$$

Similar to the fermionic or super cases,[13,31] here we consider the Hamiltonians constructed by suitable combinations of operators $a_i^n a_i^{+n}$, $a_i^{+n} a_i^n$, $(a_i a_i^+)^n$, $(a_i^+ a_i)^n$ (and N_i), $i = 1,\cdots,m$, i.e.

$$H = H[a_i^n a_i^{+n},\ a_i^{+n} a_i^n,\ (a_i a_i^+)^n,\ (a_i^+ a_i)^n,\ N_i],$$

$$JHJ = \tilde{H}, \quad (49)$$

and $E(n_1,\cdots,n_m)$ in Eq.(48) is the eigenvalue of the Hamiltonian related to the eigenstate $|n_1,\cdots,n_m\rangle$.

Now we show that, for all $k = 1,2,\cdots,\infty$, the q_k-TFD constructed above may be nicely fitted into the frame of AFSM.[17,18] Firstly, Eqs.(10) and (11) suggest that we should identify $|0(\beta)_m\rangle_k$ with Ω. Thus, our task is, in fact, to show that the above $|0(\beta)_m\rangle_k$, and Δ in Eq.(10), etc, satisfy the formulae (3)–(5), (7) and (9).

From the completeness of the basis (35) and the fact $[n_i] > 0$ for $n_i = 1,2,\cdots,k$, the separating and cyclic properties of $|0(\beta)_m\rangle_k$ follow, thus the relations (3) and (4) are fulfilled. The expressions (10) and (15) are consistent with each other because of Eq.(49). These give $J\overline{H}J = -\overline{H}$, and relation (9) is a direct consequence. Furthermore, by the definitions of $|0(\beta)_m\rangle_k$, $\overline{\theta}$ and $\overline{H}$, it is ready to verify that

$$J|0(\beta)_m\rangle_k = |0(\beta)_m\rangle_k, \quad \overline{H}|0(\beta)_m\rangle_k = 0,$$

$$\overline{\theta}|0(\beta)_m\rangle_k = |0(\beta)_m\rangle_k, \quad (50)$$

and by some tedious calculations we obtain

$$a_i\,|0(\beta)_m\rangle_k = J\mathrm{e}^{-\frac{1}{2}\beta\overline{H}} a_i^+\,|0(\beta)_m\rangle_k,$$

$$a_i^+|0(\beta)_m\rangle_k = J\mathrm{e}^{-\frac{1}{2}\beta\overline{H}} a_i|0(\beta)_m\rangle_k,$$

$$N_i|0(\beta)_m\rangle_k = J\mathrm{e}^{-\frac{1}{2}\beta\overline{H}} N_i|0(\beta)_m\rangle_k$$

$$i = 1,\cdots,m. \quad (51)$$

Owing to formulae (50), Eqs.(51) can be extended so that

$$J\mathrm{e}^{-\frac{1}{2}\beta\overline{H}} Q\,|0(\beta)_m\rangle_k = Q^+\,|0(\beta)_m\rangle_k,$$

$$\forall Q \in \mathcal{M}_{k;m}; \quad (52a)$$

and similar calculations can give

$$J\mathrm{e}^{\frac{1}{2}\beta\overline{H}} Q'\,|0(\beta)_m\rangle_k = Q'^+\,|0(\beta)_m\rangle_k,$$

$$\forall Q' := JQJ \in \mathcal{M}'_{k;m}. \quad (52b)$$

Equations (52a) and (52b) are, in fact, Eq.(5).

As for Eq.(7), we note that, from the properties of J and the explicit representation (10) for Δ, the conditions $(J1)$, $(J2)$, $(\Delta 1)$, $(\Delta 2)$ in Eq.(7) are evidently satisfied, and $(J4)$, $(\Delta 4)$ are implied by Eqs.(50). Moreover, from the separating and cyclic properties of $|0(\beta)_m\rangle_k$, $(J3)$ and $(\Delta 3)$ are true due to the Tomita–Takesaki theorem.[17] From Eqs.(50) and (52) we have

$$\begin{aligned} &{}_k\langle 0(\beta)_m|QJQJ|0(\beta)_m\rangle_k \\ =&{}_k\langle 0(\beta)_m|QJQ|0(\beta)_m\rangle_k \\ =&{}_k\langle 0(\beta)_m|Q\mathrm{e}^{-\frac{1}{2}\beta\overline{H}}Q^+|0(\beta)_m\rangle_k \geq 0, \end{aligned} \quad (53)$$

which is a representation of $(J5)$.

Finally, we consider $(\Delta 5)$. Since, when $k = \infty$, all results above are reduced to the usual TFD for a bosonic system, which has been well dealt with in Refs.[10]–[13], [17], [18], [25] and [26], here we discuss the cases of $1 \leq k < \infty$. For these cases, the operator Δ in Eq.(10) is bounded. So Δ^α with α real is well defined and invertible. Thus we have

$$\Delta^\alpha \mathcal{M}_{k;m}\Delta^{-\alpha} = \mathcal{M}_{k;m}. \quad (54)$$

Combining this with $(\Delta 3)$ gives (cf also Ref.[17])

$$\Delta^z \mathcal{M}_{k;m}\Delta^{-z} = \mathcal{M}_{k;m}, \quad (55)$$

where z is complex. Now we define the time evolution of an operator as

$$\begin{aligned} Q(t) :=& \mathrm{e}^{\mathrm{i}\overline{H}t} Q \mathrm{e}^{-\mathrm{i}\overline{H}t} \qquad \forall Q \in \mathcal{M}_{k;m} \text{ or } \tilde{\mathcal{M}}_{k;m} \\ =& \sigma_{-t/\beta}[Q], \quad \text{(cf. Eqs.(15), and (8))}. \end{aligned} \quad (56)$$

Noticing the structure of q_k-algebra $\mathcal{U}_{k;m}$, we consider, without loss of generality, an operator $A(t) = \mathrm{e}^{\mathrm{i}\overline{H}t} A \mathrm{e}^{-\mathrm{i}\overline{H}t}$, $A = c_A a_1^{+n_1} a_1^{l_1} \cdots a_i^{+n_i} a_i^{l_i} \cdots a_m^{+n_m} a_m^{l_m} \in \mathcal{U}_{k;m}$ $(0 \leq n_i,\ l_i \leq k,\ i = 1,\cdots,m)$ with c_A being a complex number, then from Eq.(51) we can obtain by some straightforward but very tedious calculations

$$A(t)\,|0(\beta)_m\rangle_k = q^{-\eta_A^2}\tilde{A}^+\left(t - \frac{\mathrm{i}}{2}\beta\right)|0(\beta)_m\rangle_k,$$

$${}_k\langle 0(\beta)_m|\,A(t) = q^{\eta_A^2}\,{}_k\langle 0(\beta)_m|\,\tilde{A}^+\left(t + \frac{\mathrm{i}}{2}\beta\right), \quad (57)$$

where we have used the symbol $\eta_A := \sum_{i=1}^m (n_i - l_i)$. Equations (57) generalize the usual thermal state condition (16) and will be called the q_k-thermal state condition. Introducing another operator $B(t') = \mathrm{e}^{\mathrm{i}\overline{H}t'} B \mathrm{e}^{-\mathrm{i}\overline{H}t'}$, $B = c_B \prod_{i=1}^m a_i^{+r_i} a_i^{s_i}$, $\eta_B = \sum_{i=1}^m (r_i - s_i)$, it can be verified that

$$\begin{aligned} &{}_k\langle 0(\beta)_m|A(t)B(t')|0(\beta)_m\rangle_k \\ =&q^{2\eta_A(\eta_A+\eta_B)}\,{}_k\langle 0(\beta)_m|B(t')A(t+\mathrm{i}\beta)|0(\beta)_m\rangle_k, \end{aligned} \quad (58)$$

where Eqs.(41) and (57) have been used. We call Eq.(58) the q_k-Kubo–Martin–Schwinger (KMS) relation. When $k=1$, it is reduced to the result given by Ref.[10]. However, for the case considered in this paper, the expression of $|0(\beta)_m\rangle_k$ in Eq.(48) implies that only if $n_i-l_i+r_i-s_i=0,\ i=1,\cdots,m$, may both sides of Eq.(58) not be zero. Therefore, we can write Eq.(58) as

$$\begin{aligned}&{}_k\langle 0(\beta)_m|A(t)B(t')|0(\beta)_m\rangle_k\\ =&{}_k\langle 0(\beta)_m|B(t')A(t+\mathrm{i}\beta)|0(\beta)_m\rangle_k,\end{aligned}\tag{59}$$

and this with Eq.(55) is precisely the KMS condition.[13,16,17] Furthermore, Eqs.(56) and (59) imply that the condition ($\Delta 5$) in Eq.(7) is satisfied.

5. Some simple applications

For simplicity, we consider a free multi-mode q_k-oscillator system whose Hamiltonian is

$$H=\sum_{i=1}^{m}\omega_i N_i,\tag{60}$$

where ω_i is the frequency (energy) of a q_k-oscillator of mode i. For this case, $E(n_1,\cdots,n_m)=\sum_{i=1}^{m}\omega_i n_i$ and from Eq.(48) it follows that

$$\begin{aligned}|0(\beta)_m\rangle_k=&Z_k^{-1/2}(\beta;m)\\ &\times\prod_{i=1}^{m}\left\{\sum_{n_i=0}^{k}\frac{(q^{-1}\mathrm{e}^{-\frac{1}{2}\beta\omega_i}a_i^+\tilde{a}_i^+)^{n_i}}{[n_i]!}\right\}\Big|0\Big\rangle,\end{aligned}\tag{61a}$$

$$Z_k(\beta;m)=\prod_{i=1}^{m}\frac{1-\mathrm{e}^{-\beta\omega_i(k+1)}}{1-\mathrm{e}^{-\beta\omega_i}}.\tag{61b}$$

Thus, the statistical averages of some operators can be obtained easily.

i. *Statistical average of number operators*

From formulae (11) and (32), we obtain

$$\begin{aligned}\langle N_i\rangle_k=&{}_k\langle 0(\beta)_m|N_i|0(\beta)_m\rangle_k\\ =&\frac{\mathrm{e}^{\beta\omega_i(k+1)}-(k+1)\mathrm{e}^{\beta\omega_i}+k}{(\mathrm{e}^{\beta\omega_i}-1)(\mathrm{e}^{\beta\omega_i(k+1)}-1)},\end{aligned}\tag{62}$$

for any $k\in\{1,2,\cdots,\infty\}$ and $i\in\{1,\cdots,m\}$. When $k=1$, this is reduced to $\langle N_i\rangle_1=\dfrac{1}{\mathrm{e}^{\beta\omega_i}+1}$, which is just the result for a usual fermionic system. When $k\to\infty$, Eq.(62) gives $\langle N_i\rangle_\infty=\dfrac{1}{\mathrm{e}^{\beta\omega_i}-1}$, the result for a bosonic system. For other cases with $2\le k<\infty$, the results are new.

ii. q_k-*statistical distributions*

When studying the statistical mechanical properties of q-deformation systems, the so-called q-statistical distribution $\langle a^+a\rangle$ is often considered (see, for example, Ref.[19]). From Eq.(29) we have

$$\begin{aligned}\langle a_i^+a_i\rangle_k=&{}_k\langle 0(\beta)_m|[N_i]|0(\beta)_m\rangle_k\\ =&\frac{(\mathrm{e}^{\beta\omega_i(k+1)}+1)(\mathrm{e}^{\beta\omega_i}-1)}{(\mathrm{e}^{\beta\omega_i(k+1)}-1)(q\mathrm{e}^{\beta\omega_i}-1)(q^{-1}\mathrm{e}^{\beta\omega_i}-1)}.\end{aligned}\tag{63}$$

When $k=1$, we have $q=i$ and Eq.(63) gives $\langle a_i^+a_i\rangle_1=\dfrac{1}{\mathrm{e}^{\beta\omega_i}+1}$. When $k\to\infty$, we have $q=1$ and Eq.(63) becomes $\langle a_i^+a_i\rangle_\infty=\dfrac{1}{\mathrm{e}^{\beta\omega_i}-1}$. We see that, in the cases $k=1$ and $k=\infty$, $\langle N_i\rangle$ and $\langle a_i^+a_i\rangle$ are the same. This is due to the fact that in the usual fermionic and bosonic cases we have $N_i=a_i^+a_i$. However, for other cases with $2\le k<\infty$, $\langle N_i\rangle_k$ and $\langle a_i^+a_i\rangle_k$ are different.

iii. q_k-*detailed balance*

From Eqs.(29), (56), (59) and (60) we have

$$\begin{aligned}\langle[N_i]\rangle_k=&{}_k\langle 0(\beta)_m|a_i^+a_i|0(\beta)_m\rangle_k\\ =&{}_k\langle 0(\beta)_m|a_ia_i^+(\mathrm{i}\beta)|0(\beta)_m\rangle_k\\ =&{}_k\langle 0(\beta)_m|a_i\mathrm{e}^{-\beta\overline{H}}a_i^+|0(\beta)_m\rangle_k\\ =&\mathrm{e}^{-\beta\omega_i}{}_k\langle 0(\beta)_m|a_ia_i^+|0(\beta)_m\rangle_k\\ =&\mathrm{e}^{-\beta\omega_i}\langle[N_i+1]\rangle_k.\end{aligned}\tag{64}$$

When $k=1$ or $k=\infty$, this becomes $\langle N_i\rangle=\mathrm{e}^{-\beta\omega_i}\times\langle N_i+1\rangle$, which is the law of detailed balance for usual statistics physics.

6. Conclusions and discussion

A type of q_k-TFD suitable for the so-called multi-mode q_k-particle systems is constructed for all $k\in\{1,2,\cdots,\infty\}$ in a unified way. We give the q_k-Pauli exclusion principle, q_k-Klein transformations, q_k-tilde operator algebras, q_k-thermal vacuum states, q_k-thermal state conditions, q_k-KMS relations and q_k-statistical average of some operators, etc. Furthermore, we demonstrate that these formalisms of q_k-TFD can all be nicely fitted into the algebraic formulation of statistical mechanics. These mean that we have obtained infinitely many realizations of the AFSM. These results contain the usual bosonic ($k=\infty$) and fermionic ($k=1$) theories as special cases. Thus, we give a kind of generalization of Ojima's discussion in Ref.[12]. When $2\le k<\infty$, the corresponding results are new. At the same time, we have constructed a type of multi-mode q_k-oscillator algebra, which is itself probably mathematically interesting.

Finally, it should be pointed out that in the discussion of this paper, the number of modes in q_k-particle systems is finite, i.e. $m < \infty$. In order to obtain a corresponding local field theory construction, it may be useful to give a Bargmann–Fock representation and the coherent states for $\mathcal{U}_{k;m}$, and then investigate the limit of $m \to \infty$. These will be considered in future work.

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