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Bifurcation and solitary waves of the nonlinear wave equation with quartic polynomial potential*

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For the nonlinear wave equation with quartic polynomial potential, bifurcation and solitary waves are investigated. Based on the bifurcation and the energy integral of the two-dimensional dynamical system satisfied by the travelling waves, it is very interesting to find different sufficient and necessary conditions in terms of the bifurcation parameter for the existence and coexistence of bright, dark solitary waves and shock waves. The method of direct integration is developed to give all types of solitary wave solutions. Our method is simpler than other newly developed ones. Some results are similar to those obtained recently for the combined KdV–mKdV equation.

Keywords: bifurcation, solitary waves, nonlinear wave equation
PACC: 0547, 0340K, 0290

1. Introduction

In the past several decades, the solitary waves of some fundamental nonlinear wave equations such as the sine-Gordon (sG), Klein–Gordon (KG), Korteweg–de Vries (KdV) and nonlinear Schrödinger (NLS) equations have been investigated extensively.[1−4] In Ref.[1], the solitary and shock waves of partial differential equations (PDEs) were viewed as the homoclinic and heteroclinic orbits of ordinary differential equations (ODEs) satisfied by the travelling waves. In Ref.[5], a one-dimensional damped system was investigated and different solitary waves were predicted. In Ref.[6], kinks in a system with degenerate critical points for the potential possessing two minimal values were discussed. For solving the exact periodic, solitary wave and multi-soliton solutions of various nonlinear PDEs, the inverse scattering method, Bäcklund transformation and bilinear form were the most used methods. In recent years, some new methods such as homogeneous balance,[7−10] hyperbolic function expansion[11−13] and elliptic function expansion[14] have also been developed. However, the mechanism for the existence of different solitary waves of nonlinear PDEs is not known completely. The newly developed methods are not simple. The method of direct integration has not been well developed to give all types of solitary waves, although it is standard. For these reasons, recently we studied the solitary waves of a perturbed sG and the combined KdV–mKdV equation based on bifurcation[15,16] and we obtained some new results.

In this paper, we apply the methods in Refs.[15] and [16] to the nonlinear wave equation with quartic polynomial potential, $U(u)=\frac{1}{4}u^4+\frac{1}{3}\alpha u^3+\frac{1}{2}\beta u^2-\lambda u$, namely,

$$u_{tt}-c_0^2u_{xx}+U'(u)=0, \tag{1}$$

where subindices indicate the second-order derivatives with respect to the corresponding variable. A vast variety of important nonlinear systems, such as the standard ϕ^4 model,[2] the combined KdV–mKdV equation,[3,16] the system with potential function possessing two minimal values,[5,6] have the form of potential function like $U(u)$ directly or indirectly. For $\alpha=0$ and $\lambda=0$, a dissipative nonlinear wave equation with quartic polynomial potential was studied and the exact solitary wave solutions were given by the homogeneous balance method.[8] In this paper, we investigate the existence and solvability of the solitary waves of Eq.(1) based on the bifurcation and the energy integral of a two-dimensional dynamical system satisfied by the travelling waves. In section 2, we analyse the bifurcation of equilibrium states by selecting λ as the bifurcation parameter. In section 3, we obtain different sufficient and necessary conditions for the ex-

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istence and coexistence of solitary or shock waves. In section 4, we give all types of solitary wave solutions by direct integration.

2. Bifurcation analysis

In this section, as a preparation for studying the solitary waves of Eq.(1), we analyse briefly the bifurcation of equilibrium states of the ODE satisfied by the travelling waves of Eq.(1).

Setting $u = u(\xi)$, $\xi = x - ct$ in Eq.(1) and denoting $u'' = \dfrac{\mathrm{d}^2 u}{\mathrm{d}\xi^2}$, we obtain

$$u'' = \frac{1}{c^2 - c_0^2}[-f(u) + \lambda], \tag{2}$$

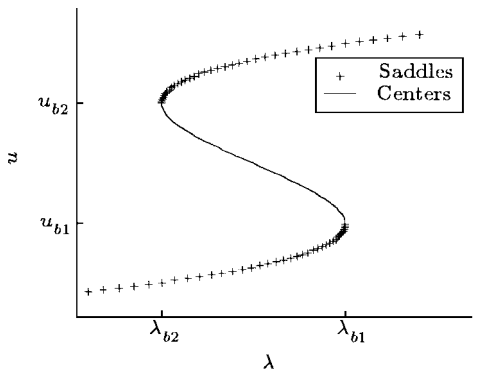
where the constant c is the wave speed, $f(u) = U'(u) + \lambda = u^3 + \alpha u^2 + \beta u$. Equation (2) describes a generalized nonlinear oscillator and can be rewritten as the following equivalent two-dimensional dynamical system

$$\begin{cases} u' = v, \\ v' = \dfrac{1}{c^2 - c_0^2}[-f(u) + \lambda], \end{cases} \tag{3}$$

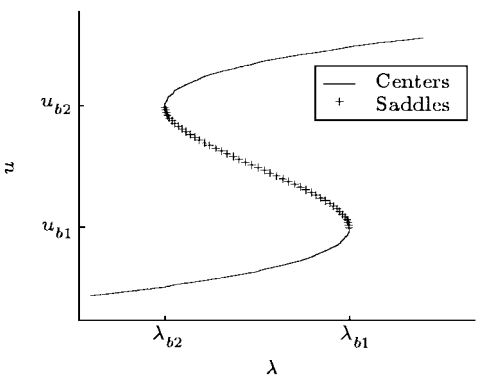
where $u' = \dfrac{\mathrm{d}u}{\mathrm{d}\xi}$, $v' = \dfrac{\mathrm{d}v}{\mathrm{d}\xi}$.

The bifurcation of equilibrium states of Eq.(3) can be analysed by selecting λ as the bifurcation parameter and generalizing the method of bifurcation analysis for the absorptive optical bistability equation.[17] If $\alpha^2 > 3\beta$ or $\Delta = \alpha^2 - 3\beta > 0$, then Eq.(3) has two bifurcation points denoted by $(u_{b1}, 0, \lambda_{b1})$ and $(u_{b2}, 0, \lambda_{b2})$, where $u_{b1} = -\dfrac{1}{3}(\lambda + \sqrt{\Delta})$, $u_{b2} = -\dfrac{1}{3}(\lambda - \sqrt{\Delta})$, $\lambda_{b1} = f(u_{b1})$ and $\lambda_{b2} = f(u_{b2})$, which are defined by $f(u) = \lambda$ and $f'(u) = 0$. Since $f'(u) = 3(u - u_{b1})(u - u_{b2})$ and $u_{b1} < u_{b2}$, we have $\lambda_{b2} < \lambda_{b1}$. Equation (3) has an infinite number of equilibrium points like $(u(\lambda), 0, \lambda)$ with $u(\lambda)$ satisfying the equation $f(u) = \lambda$, which is called the equilibrium curve. The algebraic equation $f(u) = \lambda$ has three solution branches that depend on the bifurcation parameter λ as follows: $u = u_1(\lambda)$, $u_1 > u_{b2}$ (upper branch); $u = u_2(\lambda)$, $u_{b1} < u_2 < u_{b2}$ (middle branch); and $u = u_3(\lambda)$, $u_3 < u_{b2}$ (lower branch). Furthermore, $\dfrac{\mathrm{d}u_1}{\mathrm{d}\lambda}, \dfrac{\mathrm{d}u_3}{\mathrm{d}\lambda} > 0$ and $\dfrac{\mathrm{d}u_2}{\mathrm{d}\lambda} < 0$.

If $c^2 < c_0^2$, then $(u_1, 0, \lambda)$ and $(u_3, 0, \lambda)$ are both unstable saddles, whereas $(u_2, 0, \lambda)$ is a stable centre. Hence the bifurcation diagram of Eq.(3) when $c^2 < c_0^2$ is the S-shape curve depicted in Fig.1(a). Similarly, the bifurcation diagram of Eq.(3) when $c^2 > c_0^2$ is the S-shape curve depicted in Fig.1(b). In Fig.1, the solid and '+' typed lines denote the stable centres and the unstable saddles, respectively.



(a) $c^2 < c_0^2$



(b) $c^2 > c_0^2$

Fig.1. The bifurcation diagrams of Eq.(3).

3. The existence of different solitary waves

In this section, in order to show the mechanism for the existence of different solitary waves of Eq.(1), we investigate the existence of different homoclinic and heteroclinic orbits of the ODE (3) based on the bifurcation diagram of Fig.1 and the energy integral of Eq.(3) as follows

$$\frac{1}{2}v^2 + P(u) \equiv A, \tag{4}$$

where A is the integral constant denoting energy, and $P(u)$ is the potential function

$$\begin{aligned} P(u) =& \frac{1}{c^2 - c_0^2}\left[\int_0^u f(u)\mathrm{d}u - \lambda u\right] \\ =& \frac{1}{c^2 - c_0^2}U(u). \end{aligned} \tag{5}$$

For the bifurcation parameter λ in the bistable range, we know from Fig.1 that three branches of the equilibrium solutions coexist when $\lambda_{b2} < \lambda < \lambda_{b1}$.

Hence for the bifurcation diagram Fig.1(a) and $c^2 < c_0^2$, if a heteroclinic orbit of Eq.(3) exists connecting the saddle $(u_1, 0, \lambda)$ with the saddle $(u_3, 0, \lambda)$, then there is a shock wave of Eq.(1). An important and interesting phenomenon is that there is a homoclinic orbit of Eq.(3) starting from the saddle $(u_1, 0, \lambda)$ (or $(u_3, 0, \lambda)$) and returning to the saddle $(u_1, 0, \lambda)$ (or $(u_3, 0, \lambda)$) itself. Thus there is a dark (or bright) solitary wave of Eq.(1). We emphasize here that this phenomenon depends on the bifurcation parameter λ and has not been analysed elsewhere for Eq.(1). In the following analysis, we divide the whole discussion into two cases.

*Firstly, we study the case of the bifurcation diagram Fig.*1(a) *and* $c^2 < c_0^2$. Taking $v = 0$ in the energy integral (4), we obtain the equation of the equipotential line by $P(u) = A$. Since in Fig.1(a), the integral constants of orbits of Eq.(3) passing through the saddles $(u_1, 0, \lambda)$ and $(u_3, 0, \lambda)$ are respectively

$$A_1 = P(u_1), \quad A_3 = P(u_3),$$

the corresponding equations of the equi-potential lines are $P(u) = A_1 = P(u_1)$ and $P(u) = A_3 = P(u_3)$, or respectively

$$G_1(u) = P(u) - P(u_1) = 0, \tag{6}$$

$$G_3(u) = P(u) - P(u_3) = 0, \tag{7}$$

where $G_1(u)$ and $G_3(u)$ satisfy $\lim\limits_{u \to \pm\infty} G_1(u) = -\infty$ and $\lim\limits_{u \to \pm\infty} G_3(u) = -\infty$, respectively. Equation (4) now becomes two different forms as follows

$$u' = \pm\sqrt{2[P(u_1) - P(u)]},$$

$$\text{for } u \text{ s.t. } P(u_1) \geq P(u) \text{ or } G_1(u) < 0, \tag{8}$$

$$u' = \pm\sqrt{2[P(u_3) - P(u)]},$$

$$\text{for } u \text{ s.t. } P(u_3) \geq P(u) \text{ or } G_3(u) < 0. \tag{9}$$

The existence of the solitary waves of Eq.(1) is related to the algebraic Eqs.(6) and (7). We now consider Eq.(6) mainly. On one hand, since

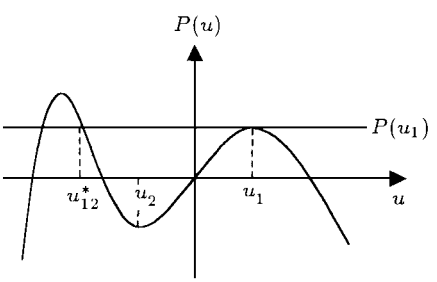
$$\begin{aligned} G_1'(u) =& \frac{1}{c^2 - c_0^2}[f(u) - \lambda] \\ =& \frac{1}{c^2 - c_0^2}(u - u_1)(u - u_2)(u - u_3) \\ & \lambda_{b2} < \lambda < \lambda_{b1}, \end{aligned}$$

we have $G_1(u_1) = 0$, $G_1'(u_1) = 0$ (i.e. Eq.(6) has a double root u_1), and $G_1'(u) > 0$ when $u_2 < u < u_1$. On the other hand, if and only if $P(u_3) > P(u_1)$, $G_1(u)$ satisfies the following two conditions:

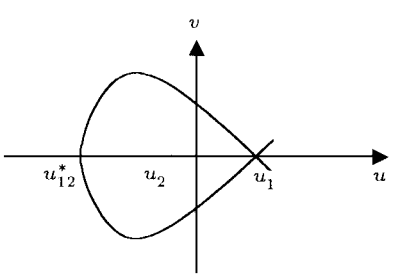
(i) $G_1(u)$ is continuous in the closed interval $[u_3\ u_2]$ and has a derivative function $G_1'(u) = (f(u) - \lambda)/(c^2 - c_0^2)$ with a fixed sign in the open interval $(u_3\ u_2)$;

(ii) $G_1(u_2) < G_1(u_1) = 0$ and $G_1(u_3) = P(u_3) - P(u_1) > 0$.

Therefore according to the well-known existence theorem of zero points of continuous functions, $G_1(u)$ has a unique zero point $u_{12}^* \in (u_3\ u_2)$, which is a single root of Eq.(6) other than the double root u_1. It is also known that $G_1(u) < 0$ or $P(u_1) \geq P(u)$ only when $u_{12}^* < u < u_1$. The function $P(u)$ can be illustrated as in Fig.2(a). By using the geometric relation between the potential function of the Hamilton system and the homoclinic orbit on the phase plane, we know that there is a unique homoclinic orbit of Eq.(3) starting from the saddle point $(u_1, 0, \lambda)$ of Fig.1(a) and returning to the saddle point $(u_1, 0, \lambda)$ itself, as illustrated in Fig.2(b). It is easily seen that u on the homoclinic orbit varies between u_1 and u_{12}^*. The existence of a unique homoclinic orbit of Eq.(3) implies that there is a unique solitary wave of Eq.(1) with the height $u_1 - u_{12}^*$ under the given condition. Moreover, the solitary wave is a dark one.



(a) The potential function.



(b) The homoclinic orbit on the phase plane.

Fig.2. The geometric relation between the potential function and the homoclinic orbit on the phase plane.

In summary, we have obtained the first sufficient and necessary condition:

Condition (I) If and only if $P(u_3) > P(u_1)$,

there exists a unique dark solitary wave of Eq.(1) with the height $u_1 - u_{12}^*$.

In the same way, we have another two conditions:

Condition (II) If and only if $P(u_3) < P(u_1)$, there is a unique bright solitary wave of Eq.(1) with the height $u_{32}^* - u_3$, where $U_{32}^* \in (u_2\ u_1)$ is unique single root of Eq.(7).

Condition (III) If and only if $P(u_3) = P(u_1)$, there is a pair of shock waves of Eq.(1) with the height $u_1 - u_3$.

It follows from condition (III) that $G_1(u_3) = G_3(u_1) = 0$. Hence $G_1(u) = G_3(u) = G(u)$. At this stage, $G(u_i) = G'(u_i) = 0$, $i = 1, 3$, or $G(u) = \frac{1}{4(c_0^2 - c^2)}(u-u_1)^2(u-u_3)^2$. This determines a critical parameter value $\lambda_{\rm cr} = -\alpha/3$ at which the bright and dark solitary waves of Eq.(1) are distinguished from the shock waves. Furthermore, in terms of the bifurcation parameter λ, the three sufficient and necessary conditions can be rewritten respectively as follows.

Condition (I)' If and only if $\lambda_{b2} < \lambda < \lambda_{\rm cr}$, there is a unique dark solitary wave of Eq.(1) with the height $u_1 - u_{12}^*$.

Condition (II)' If and only if $\lambda_{\rm cr} < \lambda < \lambda_{b1}$, there is a unique bright solitary wave of Eq.(1) with the height $u_{32}^* - u_3$.

Condition (III)' If and only $\lambda = \lambda_{\rm cr}$, there is a pair of shock waves of Eq.(1) with the height $u_1 - u_3$.

*Secondly, we study the other case of the bifurcation diagram Fig.*1(b) *and* $c^2 > c_0^2$. We can also obtain three different conditions for the existence of solitary waves of Eq.(1) by modifying the previous discussion. Thus the results are listed without detailed analyses. The equation of the equi-potential line of the orbit passing through the saddle $(u_2, 0)$ is

$$G_2(u) = P(u) - P(u_2) = 0, \tag{10}$$

where $G_2(u)$ satisfies the condition $\lim\limits_{u\to\pm\infty} G_2(u) = +\infty$. Equation (8) has a double root u_2. We have

Condition (IV) If and only if $\lambda_{b2} < \lambda < \lambda_{b1}$, a bright solitary wave of Eq.(1) with the height $u_{21}^* - u_2$ and a dark solitary wave of Eq.(1) with the height $u_2 - u_{23}$ coexist, where u_{21}^* and u_{23}^* are two single roots of Eq.(10) satisfying $u_{23}^* < u_3 < u_2 < u_1 < u_{21}^*$.

Condition (V) If and only if $\lambda = \lambda_{b1}$, there is a unique bright solitary wave of Eq.(1) with the height $u_{b1}^* - u_{b1}$, where u_{b1} is a triple root of Eq.(10), u_{b1}^* is unique single root of Eq.(10) satisfying $u_{b1}^* > u_1$.

Condition (VI) If and only if $\lambda = \lambda_{b2}$, there is a unique dark solitary wave of Eq.(1) with the height $u_{b2} - u_{b2}^*$, where u_{b2} is a triple root of Eq.(10), u_{b2}^* is unique single root of Eq.(10) satisfying $u_{b2}^* < u_3$.

We have seen that the mechanism for the existence of different solitary waves of Eq.(1) is quite clear on the basis of the bifurcation diagram Fig.1 and the energy integral (4). The simplest condition (III)' is just the well-known case (usually α=0, $\lambda_{\rm cr} = 0$). Actually, these interesting conditions modify those possible ones of Ref.[6].

4. The solitary wave solutions by direct integration

In some simple cases, the solitary wave solutions may be given in the limiting cases of the periodic wave solutions expressed by the elliptic functions (see Refs.[1] and [3] for examples). However, in most cases, in seeking the solitary wave solutions of PDEs we encounter mathematical difficulties, and a large variety of analytic methods have to be adopted.[7−16] In this section, we give briefly all types of solitary wave solutions of Eq.(1) using a direct integration method similar to Ref.[16], in which one can find more details for the combined KdV–mKdV equation.

From the energy integral (4), and the further qualitative results and some notations in section 3, we conclude that different solitary waves of Eq.(1) must satisfy different forms of Eq.(4).

(i) If $c^2 < c_0^2$ and $\lambda_{b2} < \lambda < \lambda_{\rm cr}$, except for the double root u_1, Eq.(6) also has another single root u_{13}^* satisfying $u_{13}^* < u_3$. Hence

$$P(u_1) - P(u) = \frac{1}{4(c_0^2 - c^2)}(u_1 - u)^2(u - u_{12}^*)(u - u_{13}^*),$$

by Eq.(8), and the dark solitary wave of Eq.(1) now satisfies the ODE

$$u' = \pm m_1(u_1 - u)\sqrt{(u - u_{12}^*)(u - u_{13}^*)}$$

$$u_{12}^* < u < u_1, \tag{11}$$

where $m_1 = \frac{1}{\sqrt{2(c_0^2 - c^2)}}$. Integrating Eq.(11) yields

$$u = \frac{u_1(u_{13}^* - u_{12}^*) + u_{12}^*(u_1 - u_{13}^*)\mathrm{sech}^2\left[\frac{1}{2}m_1\sqrt{(u_1 - u_{12}^*)(u_1 - u_{13}^*)}(\xi - \xi_0)\right]}{u_{13}^* - u_{12}^* + (u_1 - u_{13}^*)\mathrm{sech}^2\left[\frac{1}{2}m_1\sqrt{(u_1 - u_{12}^*)(u_1 - u_{13}^*)}(\xi - \xi_0)\right]}, \tag{12}$$

where ξ_0 is determined by the initial value $u_0 = u(0)$. It holds that $u \to u_1$ as $\xi \to \pm\infty$.

(ii) If $c^2 < c_0^2$ and $\lambda_{\rm cr} < \lambda < \lambda_{b1}$, Eq.(7) also has another single root $u_{31}^* > u_1$. By Eq.(9), the bright solitary wave of Eq.(1) satisfies the equation

$$u' = \pm m_1(u-u_3)\sqrt{(u-u_{32}^*)(u-u_{31}^*)}$$
$$u_3 < u < u_{32}^*. \tag{13}$$

Integrating Eq.(13) yields

$$u = \frac{u_3(u_{32}^*-u_{31}^*) + u_{32}^*(u_{31}^*-u_3)\mathrm{sech}^2\left[\frac{1}{2}m_1\sqrt{(u_{31}^*-u_3)(u_{32}^*-u_3)}(\xi-\xi_0)\right]}{u_{32}^*-u_{31}^*+(u_{31}^*-u_3)\mathrm{sech}^2\left[\frac{1}{2}m_1\sqrt{(u_{31}^*-u_3)(u_{32}^*-u_3)}(\xi-\xi_0)\right]}, \tag{14}$$

where ξ_0 is determined by the initial value $u_0 = u(0)$. It holds that $u \to u_3$ as $\xi \to \pm\infty$.

(iii) If $c^2 < c_0^2$ and $\lambda = \lambda_{\rm cr}$, the shock waves of Eq.(1) satisfy the equation

$$u' = \pm m_1(u_1-u)(u-u_3) \quad u_3 < u < u_1. \tag{15}$$

Integrating Eq.(15) yields

$$u = \frac{u_3 + u_1 d_3 \exp[\pm m_1(u_1-u_3)\xi]}{1 + d_3\exp[\pm m_1(u_1-u_3)\xi]}, \tag{16}$$

where $d_3 = \dfrac{u_0-u_3}{u_1-u_0}$, and $u_0 = u(0)$ is the initial value. Equation (16) is also known as a pair of kink and anti-kink solutions of Eq.(1).

(iv) If $c^2 > c_0^2$ and $\lambda_{b2} < \lambda < \lambda_{b1}$, the solitary waves of Eq.(1) satisfy the equation

$$u' = \pm\sqrt{2[P(u_2)-P(u)]} = \pm m_2|u-u_2|\sqrt{(u_{21}^*-u)(u-u_{23}^*)}$$
$$u_{23}^* < u < u_{21}^*, \tag{17}$$

where $m_2 = \dfrac{1}{\sqrt{2(c^2-c_0^2)}}$. Integrating Eq.(17) yields the bright solitary wave of Eq.(1) as

$$u = \frac{u_2(u_{21}^*-u_{23}^*) - u_{21}^*(u_2-u_{23}^*)\mathrm{sech}^2\left[\frac{1}{2}m_2\sqrt{(u_{21}^*-u_2)(u_2-u_{23}^*)}(\xi-\xi_0)\right]}{u_{21}^*-u_{23}^*-(u_2-u_{23}^*)\mathrm{sech}^2\left[\frac{1}{2}m_2\sqrt{(u_{21}^*-u_2)(u_2-u_{23}^*)}(\xi-\xi_0)\right]}, \tag{18}$$

and the dark solitary wave of Eq.(1) as

$$u = \frac{u_2(u_{21}^*-u_{23}^*) - u_{23}^*(u_{21}^*-u_2)\mathrm{sech}^2\left[\frac{1}{2}m_2\sqrt{(u_{21}^*-u_2)(u_2-u_{23}^*)}(\xi-\xi_0)\right]}{u_{21}^*-u_{23}^*-(u_{21}^*-u_2)\mathrm{sech}^2\left[\frac{1}{2}m_2\sqrt{(u_{21}^*-u_2)(u_2-u_{23}^*)}(\xi-\xi_0)\right]}. \tag{19}$$

It holds that $u \to u_2$ as $\xi \to \pm\infty$.

(v) If $c^2 > c_0^2$ and $\lambda = \lambda_{b1}$, the bright solitary wave of Eq.(1) satisfies the equation

$$u' = \pm m_2(u-u_{b1})\sqrt{(u-u_{b1})(u_{b1}^*-u)}$$
$$u_{b1} < u < u_{b1}^*. \tag{20}$$

Integrating Eq.(20) yields

$$u = \frac{u_{b1}^* + u_{b1}\left[\frac{1}{2}m_2(u_{b1}^*-u_{b1})(\xi-\xi_0)\right]^2}{1+\left[\frac{1}{2}m_2(u_{b1}^*-u_{b1})(\xi-\xi_0)\right]^2}, \tag{21}$$

where ξ_0 is determined by the initial value $u_0 = u(0)$. It holds that $u \to u_{b1}$ as $\xi \to \pm\infty$.

(vi) If $c^2 > c_0^2$ and $\lambda = \lambda_{b2}$, the dark solitary wave of Eq.(1) satisfies the equation

$$u' = \pm m_2(u_{b1}-u)\sqrt{(u-u_{b2}^*)(u_{b2}-u)}$$
$$u_{b2}^* < u < u_{b2}. \tag{22}$$

Integrating Eq.(22) yields

$$u = \frac{u_{b2}^* + u_{b2}\left[\frac{1}{2}m_2(u_{b2}-u_{b2}^*)(\xi-\xi_0)\right]^2}{1+\left[\frac{1}{2}m_2(u_{b2}-u_{b2}^*)(\xi-\xi_0)\right]^2}, \tag{23}$$

where ξ_0 is determined by the initial value $u_0 = u(0)$. It holds that $u \to u_{b2}$ as $\xi \to \pm\infty$.

Our conclusions here are not easily derived by other new methods,[7−14] thus the method of direct integration for solving the solitary wave solutions is simpler. It is a fact that some results are similar to those obtained recently for the combined KdV–mKdV equation.[16]

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