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New multi-soliton solutions and travelling wave solutions of the dispersive long-wave equations

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Using the extended homogeneous balance method, the (1+1)-dimensional dispersive long-wave equations have been solved. Starting from the homogeneous balance method, we have obtained a nonlinear transformation for simplifying a dispersive long-wave equation into a linear partial differential equation. Usually, we can obtain only a type of soliton-like solution. In this paper, we have further found some new multi-soliton solutions and exact travelling solutions of the dispersive long-wave equations from the linear partial equation.

Keywords: soliton solution, dispersive long-wave equation, extended homogeneous balance method
PACC: 0365, 0340

1. Introduction

Nonlinear partial differential equations are widely used to describe complex phenomena in various sciences, especially in physical science. Solving nonlinear partial differential equations is an important work in nonlinear science. It is well known that there are infinitely many solutions for every nonlinear partial differential equation, and it is a difficult task to find an exact solution. A vast variety of methods including inverse scattering theory,[1] Bäcklund transformation,[2] Hirota's bilinear method,[3] Darboux transformation,[4] tanh-function method[5] and the standard truncated Painlevé analysis approach[6] have been developed to obtain travelling solutions to nonlinear partial differential equations. Recently, the homogeneous balance method,[7–9] which is an effective method for finding exact travelling wave solutions of nonlinear partial differential equations, has been developed first by Wang and has been generalized to find the Auto-Bäcklund[10,11] multi-solitary wave solutions.[8,12–14] In this paper, we still use the extended homogeneous balance method to find some new exact travelling wave solutions of the dispersive long-wave equations[15]

$$u_t + \eta_x + \frac{1}{2}(u)_x^2 = 0, \tag{1}$$

$$\eta_t + (u\eta + u + u_{xx})_x = 0, \tag{2}$$

where $\eta(x,t)$ is the amplitude of a surface wave, propagating along the x-axis with a horizontal velocity $u(x,t)$.

2. Multi-soliton solutions of the dispersive long-wave equations

Zhang[8] has obtained a Bäcklund transformation for Eqs.(1) and (2) in the form

$$u = f(\phi)_x + a, \tag{3}$$

$$\eta = f(\phi)_{xx} - 1, \tag{4}$$

where $f(\phi)$ and $\phi(x,t)$ are suitable functions, and a is a constant. These can simplify Eqs.(1) and (2) into a linear partial differential equation as follows

$$\phi_t + a\phi_x + \phi_{xx} = 0, \tag{5}$$

whereas $f(\phi)$ satisfies

$$f(\phi) = 2\ln\phi. \tag{6}$$

A type of special multi-soliton-like solution of Eqs.(1) and (2)

$$\phi(x,t) = 1 + \sum_{i=1}^{n} \exp[k_i x - (ak_i + k_i^2)t] \tag{7}$$

is also given in Ref.[8].

In fact, taking the variable transformation $x - at = \xi$, Eq.(5) becomes a standard heat conduction equation

$$\phi_t = -\phi_{\xi\xi}. \tag{8}$$

To find some multi-soliton solutions of the dispersive long-wave equation, we can take the solutions of the heat conduction equation as

$$\begin{aligned}\phi_1(x,t) =& 1+\sum_{i=1}^{n} A_i \exp[k_i x-(ak_i+k_i^2)t] \\ &+\sum_{i=1}^{m} B_i \cos[P_i x-P_i(a+2p_i)t+\xi_{i0}] \\ &\times \exp[p_i x-(p_i^2+ap_i-P_i^2)t] \\ &+\sum_{i=1}^{m} C_i \cosh[Q_i x-Q_i(a+2q_i)t+\zeta_{i0}] \\ &\times \exp[q_i x-(q_i^2+aq_i+Q_i^2)t], \end{aligned} \tag{9}$$

where $k_i, p_i, P_i, q_i, Q_i, A_i, B_i, C_i$ are arbitrary constants; n, m, l are arbitrary positive integers.

The corresponding multi-soliton solutions of the dispersive long-wave equation are given by solutions (3) and (4) with ϕ being determined by Eq.(9).

Now we discuss some special soliton solutions and the interaction properties among multi-solitons.

2.1. Single-soliton solution

If we select $A_1 = 1$, $B_i = C_i = 0$, $n = 1$ in Eq.(9), then the amplitude of a surface wave η is a single bell-soliton solution

$$\eta = \frac{k_1^2}{2}\mathrm{sech}^2\left[\frac{k_1}{2}(x-k_1 t-at)\right]-1, \tag{10}$$

where k_1, a are arbitrary constants. While the horizontal velocity u is a single kink soliton solution

$$u = k_1 \tanh\frac{k_1(x-k_1 t-at)}{2}+k_1+a. \tag{11}$$

Figures 1 and 2 show the related single bell soliton for η and the kink soliton for u respectively, with the conditions $k_1 = \frac{1}{2}, a = 0, A_1 = 1$.

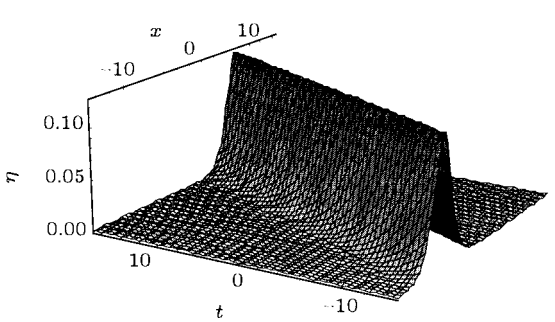


Fig.1. Evolution of single bell soliton for the field η.

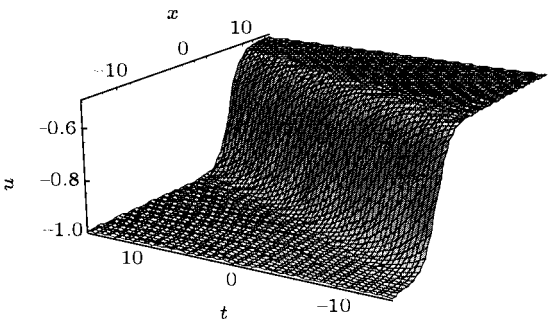


Fig.2. Evolution of single kink soliton for the field u.

2.2. Soliton fission

Usually, one believes that the interactions among (1+1)-dimensional solitons are elastic. However, in some special cases, the interactions among (1+1)-dimensional solitons may also be completely non-elastic. In some cases, the resonance among solitons may occur such that they become only one peak. And the resonance soliton may also be split at some suitable time. This phenomenon can be easily illustrated from the exact solutions (3) and (4) with Eq.(9) for the (1+1)-dimensional dispersive long-wave equation.

In general, if $B_i \neq 0$ in Eq.(9), the corresponding solutions (3) and (4) are only a singular solution, which will not be discussed. For $B_i = 0$, solution (9) denotes a fission procedure for a resonant soliton. Figure 3 shows a fission procedure of two-bell-soliton solutions for the amplitude of a surface wave η, under the conditions

$$\begin{aligned} &B_i = C_i = a = 0, \quad n = 2, \\ &A_1 = A_2 = 1, \quad k_1 = \frac{2}{5}, \quad k_2 = -\frac{2}{5}. \end{aligned} \tag{12}$$

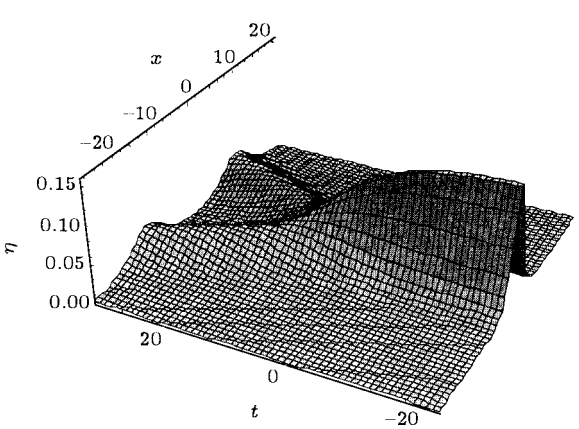


Fig.3. Two-bell-soliton fission for the field η.

From Fig.3, we see that, when $t < 0$, solution (11) with Eq.(12) is only a bell-soliton solution, a resonant soliton. As time increases to $t > 0$, the resonant soliton is fissioned to a two-soliton solution. Similarly, for the horizontal velocity u, solutions (3) and (4) with Eq.(9) denote a kink soliton which is fissioned to two kink solitons at time $t = 0$.

2.3. Soliton fusion

It is natural that soliton fusion, an inverse procedure of soliton fission, may also be found because of the integrability of the (1+1)-dimensional dispersive long-wave equation. Actually, if we take $A_i = B_i = 0$ in Eq.(9), then we can obtain some types of soliton fusion solution. For instance, Fig.4 is a plot of the two kink soliton fusion procedure for the horizontal velocity u, under the condition

$$A_i = B_i = a = 0, \quad l = 1,$$
$$C_1 = 1, \quad q_1 = \frac{1}{3}, \quad Q_1 = \frac{4}{3}. \tag{13}$$

When $t < 0$, there are two kink solitons for the horizontal velocity u given by solutions (3) and (4) with Eq.(9). As t increases over zero, two kink solitons are fused to a single resonant kink.

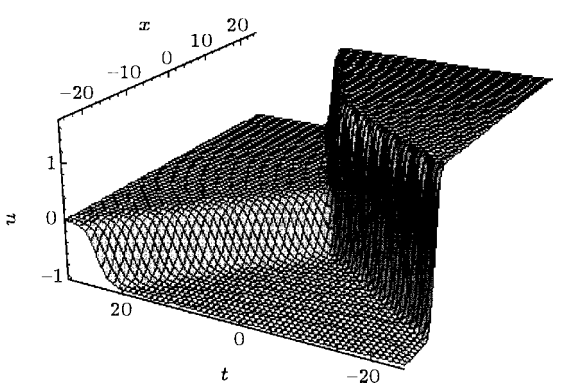


Fig.4. Two kink soliton fusion for the field u.

3. Other travelling wave solutions

Case 1

We let

$$\phi_2(x,t) = \sum_{i=1}^{n}(A_i x + B_i t + C_i), \tag{14}$$

and inserting Eq.(14) into Eq.(5), we obtain

$$B_i = -aA_i - A_i^2, \tag{15}$$

where A_i and C_i are two arbitrary constants. So we can obtain

$$\phi_2(x,t) = \sum_{i=1}^{n}(A_i x - (aA_i + A_i^2)t + C_i). \tag{16}$$

Case 2

We let

$$\phi_3(x,t) = \sum_{i=1}^{n}[A_i \exp(k_i x + \omega_i t + \eta_{i0}) + B_i], \tag{17}$$

then putting Eq.(17) into Eq.(5), we have

$$\omega_i = -k_i^2 - ak_i, \tag{18}$$

where A_i, B_i, k_i and η_{i0} are all arbitrary constants. So we have

$$\phi_3(x,t) = \sum_{i=1}^{n}[A_i \exp(k_i x - (k_i^2 + ak_i)t + \eta_{i0}) + B_i]. \tag{19}$$

Case 3

We let

$$\phi_4(x,t) = A(t)\sin x + B(t)\cos x. \tag{20}$$

Then by inserting Eq.(20) into Eq.(5), we deduce

$$A_t(t) - A(t) - aB(t) = 0, \tag{21}$$

$$B_t(t) - B(t) - aA(t) = 0, \tag{22}$$

of which the general solutions are

$$A(t) = \exp t(c_1 \sin at + c_2 \cos at), \tag{23}$$

$$B(t) = \exp t(c_2 \cos at - c_1 \sin at). \tag{24}$$

Hence we have

$$\phi_4(x,t) = \exp(t)\{[c_1 \sin at + c_2 \cos at]\sin x + [c_2 \cos at + c_1 \sin at]\cos x\}. \tag{25}$$

Case 4

We let

$$\phi_5(x,t) = A(x)t + B(x), \tag{26}$$

and, putting Eq.(26) into Eq.(5), we find

$$A_{xx}(x) + aA_x(x) = 0, \tag{27}$$

$$B_{xx}(t) + aB_x(x) + A(x) = 0, \tag{28}$$

of which the general solutions are

$$A(x) = c_1 + c_2 \exp(-ax), \tag{29}$$

$$B(x) = c_3 - \frac{c_1}{a}x + \left(\frac{c_2 x}{a^2} - \frac{c_4}{a}\right)\exp(-ax). \quad (30)$$

Thus we obtain

$$\phi_5(x,t) = [c_1 + c_2\exp(-ax)]t + c_3 - \frac{c_1}{a}x + \left(\frac{c_2 x}{a^2} - \frac{c_4}{a}\right)\exp(-ax). \quad (31)$$

The corresponding multi-soliton solutions of the dispersive long-wave equation is given by solutions (3) and (4) with ϕ being determined by Eq.(9).

4. Summary and discussion

In this paper, by using the extended homogeneous balance method, a Bäcklund transformation of the (1+1)-dimensional dispersive long-wave equations is first obtained. Then Eqs.(1) and (2) are transformed into a (1+1)-dimensional heat conduction equation. In terms of a (1+1)-dimensional heat conduction equation, a quite general solution which includes both multiple singular and nonsingular solitons is obtained. Moreover, some new travelling wave solutions are also given.

Usually, it is considered that the interactions among solitons are completely elastic. However, in some special cases, the soliton interactions may be completely non-elastic. From the above, we see that the soliton fission and fusion procedures may occur for the (1+1)-dimensional dispersive long-wave equations.

Because of the wide applications of soliton theory in real physics, it is worthwhile to study soliton fission and fusion further.

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