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Algebraic structure and Poisson integrals of a rotational relativistic Birkhoff system*

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We have studied the algebraic structure of the dynamical equations of a rotational relativistic Birkhoff system. It is proven that autonomous and semi-autonomous rotational relativistic Birkhoff equations possess consistent algebraic structure and Lie algebraic structure. In general, non-autonomous rotational relativistic Birkhoff equations possess no algebraic structure, but a type of special non-autonomous rotational relativistic Birkhoff equation possesses consistent algebraic structure and consistent Lie algebraic structure. Then, we obtain the Poisson integrals of the dynamical equations of the rotational relativistic Birkhoff system. Finally, we give an example to illustrate the application of the results.

Keywords: rotational relativity, Birkhoff equation, algebraic structure, Poisson integral
PACC: 0316, 0320, 0220

1. Introduction

In 1979, Bengtsson and Frauendorf [1] accurately measured the maximum values of spin velocity of fourteen types of nucleon, and the results showed that these maximum values were different from each other. With the development of science and technology, more and more experimental results are related to high-speed spinning. In 1985, Carmeli[2−5] constructed the theory of rotational relativistic mechanics. Since 1996, we have constructed the theory of rotational relativistic analytic mechanics.[6−13] Birkhoff dynamics is more general than Hamilton mechanics.[14−26] Hamilton mechanics is diffusely applied in the field of modern physics, so Birkhoff dynamics should play an important role in the field of modern physics. Recently, we have constructed the basic theory of dynamics of the rotational relativistic Birkhoff system, and we have discussed its relations with Lagrange mechanics and Hamilton mechanics of the rotational relativistic system in Ref.[27]. Furthermore, we have given the geometrical and symmetrical theories and the integral method of dynamics of the rotational relativistic Birkhoff system in Refs.[28]–[31]. In 1999, the algebraic structures of classical and quantum systems were studied by Yang and Hou.[32,33] In this paper, the algebraic structure and Poisson integrals of a rotational relativistic Birkhoff system are studied. Firstly, it is proven that autonomous and semi-autonomous rotational relativistic Birkhoff equations possess consistent algebraic structure and Lie algebraic structure. In general, non-autonomous rotational relativistic Birkhoff equations possess an algebraic structure, but a type of special non-autonomous rotational relativistic Birkhoff equation possesses consistent algebraic structure and consistent Lie algebraic structure. Then the Poisson integrals of the dynamical equations of the rotational relativistic Birkhoff system are obtained. Finally, an example to illustrate the application of the results is given.

2. Birkhoff equations of a rotational relativistic system

We study the fixed-axis rotation of a mechanical system of N particles. At time t, the angular velocity of the ith particle is $\dot{\theta}_i$, the limiting angular velocity is Γ_i, the classical rotational moment of inertia is I_{0i}, and its relativistic rotational moment of inertia is[5−7]

$$I_i = \frac{I_{0i}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}}. \tag{1}$$

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Constructing the Birkhoffian B^* and the Birkhoff functions $R^*_\nu(\nu=1,\cdots,2n)$ of a rotational relativistic system

$$\begin{aligned} &B^*=B^*(I_i(t,a^\mu),t,a^\mu),\\ &R^*_\nu=R^*_\nu(I_i(t,a^\mu),t,a^\mu)\\ &(\mu,\nu=1,\cdots,2n;i=1,\cdots,N), \end{aligned} \tag{2}$$

and letting

$$\begin{aligned} &\widetilde{B}^*=\widetilde{B}^*(t,a^\mu)=B^*(I_i(t,a^\mu),t,a^\mu),\\ &\widetilde{R}^*_\nu=\widetilde{R}^*_\nu(t,a^\mu)=R^*_\nu(I_i(t,a^\mu),t,a^\mu), \end{aligned} \tag{3}$$

the Birkhoff equations of the rotational relativistic system can be written in the forms[27,28]

$$\widetilde{\Omega}^*_{\mu\nu}\dot{a}_\nu-\frac{\partial\widetilde{B}^*}{\partial a^\mu}-\frac{\partial\widetilde{R}^*_\mu}{\partial t}=0\quad(\mu,\nu=1,\cdots,2n) \tag{4}$$

or

$$\dot{a}^\mu=\widetilde{\Omega}^{*\mu\nu}\left(\frac{\partial\widetilde{B}^*}{\partial a^\nu}+\frac{\partial\widetilde{R}^*_\nu}{\partial t}\right)\quad(\mu,\nu=1,\cdots,2n), \tag{5}$$

where the repeated subscripts express making sum, and Eqs.(5) are the contravariant algebraic forms of Eqs.(4). $\widetilde{\Omega}^*_{\mu\nu}$ is the Birkhoff tensor of the rotational relativistic system, and $\widetilde{\Omega}^{*\mu\nu}$ is the contravariant tensor of $\widetilde{\Omega}^*_{\mu\nu}$, i.e.

$$\widetilde{\Omega}^*_{\mu\nu}=\frac{\partial\widetilde{R}^*_\nu}{\partial a^\mu}-\frac{\partial\widetilde{R}^*_\mu}{\partial a^\nu},\quad \widetilde{\Omega}^{*\mu\nu}=\left(\left\|\frac{\partial\widetilde{R}^*_\beta}{\partial a^\alpha}-\frac{\partial\widetilde{R}^*_\alpha}{\partial a^\beta}\right\|\right)^{\mu\nu}$$
$$(\mu,\nu,\alpha,\beta=1,\cdots,2n). \tag{6}$$

In general, we suppose that system (4) is non-singular, i.e.

$$\det(\widetilde{\Omega}^*_{\mu\nu})\neq 0. \tag{7}$$

If $\widetilde{R}^*_\mu$ and $\widetilde{B}^*$ do not include time t explicitly, the system is autonomous; if $\widetilde{R}^*_\mu$ do not include time t explicitly, the system is semi-autonomous. For the autonomous and semi-autonomous systems, the rotational relativistic Birkhoff equations (4) and (5) have the simple forms as follows

$$\widetilde{\Omega}^*_{\mu\nu}\dot{a}^\nu-\frac{\partial\widetilde{B}^*}{\partial a^\mu}=0\quad(\mu,\nu=1,\cdots,2n), \tag{8}$$

$$\dot{a}^\mu-\widetilde{\Omega}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\mu}=0\quad(\mu,\nu=1,\cdots,2n). \tag{9}$$

3. Algebraic structure of a rotational relativistic Birkhoff system

For the autonomous and semi-autonomous Birkhoff equations (9) of a rotational relativistic system, we differentiate a function $\widetilde{A}^*(a^\mu)$ with respect to t using Eqs.(9), then we define an algebraic product

$$\dot{\widetilde{A}}^*=\frac{\partial\widetilde{A}^*}{\partial a^\mu}\widetilde{\Omega}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\nu}=\widetilde{A}^*\circ\widetilde{B}^*. \tag{10}$$

The product satisfies the right distributive, left distributive and scalar laws

$$\widetilde{A}^*\circ(\widetilde{B}^*+\widetilde{C}^*)=\widetilde{A}^*\circ\widetilde{B}^*+\widetilde{A}^*\circ\widetilde{C}^*, \tag{11}$$

$$(\widetilde{A}^*+\widetilde{B}^*)\circ\widetilde{C}^*=\widetilde{A}^*\circ\widetilde{C}^*+\widetilde{B}^*\circ\widetilde{C}^*, \tag{12}$$

$$(\lambda\widetilde{A}^*)\circ\widetilde{B}^*=\widetilde{A}^*\circ(\lambda\widetilde{B}^*)=\lambda(\widetilde{A}^*\circ\widetilde{B}^*)$$
$$(\lambda\text{ is a constant}). \tag{13}$$

An algebraic function satisfying Eqs.(11)–(13) is consistent algebraic. So we have:

Conclusion 1. The Birkhoff equations of the autonomous and semi-autonomous rotational relativistic systems possess consistent algebraic structure.

Moreover, the product (10) satisfies anti-symmetries and Jacobi identity

$$\widetilde{A}^*\circ\widetilde{B}^*+\widetilde{B}^*\circ\widetilde{A}^*=0, \tag{14}$$

$$\widetilde{A}^*\circ(\widetilde{B}^*\circ\widetilde{C}^*)+\widetilde{B}^*\circ(\widetilde{C}^*\circ\widetilde{A}^*)+\widetilde{C}^*(\widetilde{A}^*\circ\widetilde{B}^*)=0. \tag{15}$$

An algebraic function satisfying Eqs.(14) and (15) is Lie algebraic. So we have:

Conclusion 2. The Birkhoff equations of the autonomous and semi-autonomous rotational relativistic systems possess a Lie algebraic structure.

We can define a generalized Poisson bracket $[\widetilde{A}^*,\widetilde{B}^*]$ using Eq.(10)

$$\frac{\partial\widetilde{A}^*}{\partial a^\mu}\widetilde{\Omega}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\nu}\overset{\text{def}}{=}[\widetilde{A}^*,\widetilde{B}^*], \tag{16}$$

which satisfies the Lie algebraic axiom

$$[\widetilde{A}^*,\widetilde{B}^*]+[\widetilde{B}^*,\widetilde{A}^*]=0, \tag{17}$$

$$\begin{aligned} &[[\widetilde{A}^*,\widetilde{B}^*],\widetilde{C}^*]+[[\widetilde{B}^*,\widetilde{C}^*],\widetilde{A}^*]\\ &+[[\widetilde{C}^*,\widetilde{A}^*],\widetilde{B}^*]=0. \end{aligned} \tag{18}$$

For the non-autonomous Birkhoff equations (5) of the rotational relativistic system, we differentiate a function $\widetilde{A}^*(a^\mu)$ with respect to t by Eqs.(5), then we define an algebraic product

$$\dot{\widetilde{A}}^*(a^\mu)=\frac{\partial\widetilde{A}^*}{\partial a^\mu}\widetilde{\Omega}^{*\mu\nu}\left(\frac{\partial\widetilde{B}^*}{\partial a^\nu}+\frac{\partial\widetilde{R}^*_\nu}{\partial t}\right)\overset{\text{def}}{=}\widetilde{A}^*\cdot\widetilde{B}^*. \tag{19}$$

We have

$$\widetilde{A}^* \cdot (\widetilde{B}^* + \widetilde{C}^*) = \widetilde{A}^* \cdot \widetilde{B}^* + \frac{\partial \widetilde{A}^*}{\partial a^\mu} \widetilde{\Omega}^{*\mu\nu} \frac{\partial \widetilde{C}^*}{\partial a^\nu} \neq \widetilde{A}^* \cdot \widetilde{B}^* + \widetilde{A}^* \cdot \widetilde{C}^*, \tag{20}$$

$$(\widetilde{A}^* + \widetilde{B}^*) \cdot \widetilde{C}^* = \widetilde{A}^* \cdot \widetilde{C}^* + \widetilde{B}^* \cdot \widetilde{C}^*, \tag{21}$$

$$(\lambda \widetilde{A}^*) \cdot \widetilde{B}^* \neq (\widetilde{A}^* \cdot \widetilde{B}^*)\lambda. \tag{22}$$

Hence the product $\widetilde{A}^* \cdot \widetilde{B}^*$ does not express an algebraic structure. The non-autonomous Birkhoff equations (5) of the rotational relativistic system do not possess a consistent algebraic structure expressed by the product (19).

We study a type of special non-autonomous Birkhoff equations of the rotational relativistic system. If the Birkhoffian and the Birkhoff functions of the rotational relativistic system in Eqs.(5) satisfy the following equation

$$\widetilde{T}^{*\mu\nu} \frac{\partial \widetilde{B}^*}{\partial a^\nu} = \widetilde{\Omega}^{*\mu\nu} \frac{\partial \widetilde{R}^*_\nu}{\partial t}, \tag{23}$$

where

$$(\widetilde{T}^{*\mu\nu}) = \begin{pmatrix} T^{11} & 0 & \cdots & 0 \\ 0 & T^{22} & 0 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & T^{2n2n} \end{pmatrix} \tag{24}$$

is the symmetrical tensor of the rotational relativistic system, and is determined by Eqs.(23). Then the non-autonomous Birkhoff equations (5) of the rotational relativistic system can be expressed in the following form

$$\dot{a}^\mu - \widetilde{S}^{*\mu\nu} \frac{\partial \widetilde{B}^*}{\partial a^\nu} = 0, \tag{25}$$

where

$$\widetilde{S}^{*\mu\nu} = \widetilde{\Omega}^{*\mu\nu} + \widetilde{T}^{*\mu\nu}. \tag{26}$$

Differentiate a function $\widetilde{A}(a^\mu)$ with respect to t using Eqs.(25), we then define a product

$$\dot{\widetilde{A}}^*(a^\mu) = \frac{\partial \widetilde{A}^*}{\partial a^\mu} \widetilde{S}^{*\mu\nu} \frac{\partial \widetilde{B}^*}{\partial a^\nu} \stackrel{\text{def}}{=} \widetilde{A}^* \widetilde{B}^*. \tag{27}$$

The product satisfies the right distributive, left distributive and scalar laws

$$\widetilde{A}^*(\widetilde{B}^* + \widetilde{C}^*) = \widetilde{A}^* \widetilde{B}^* + \widetilde{A}^* \widetilde{C}^*, \tag{28}$$

$$(\widetilde{A}^* + \widetilde{B}^*)\widetilde{C}^* = \widetilde{A}^* \widetilde{C}^* + \widetilde{B}^* \widetilde{C}^*, \tag{29}$$

$$(\lambda \widetilde{A}^*)\widetilde{B}^* = \widetilde{A}^*(\lambda \widetilde{B}^*) = \lambda(\widetilde{A}^* \widetilde{B}^*) \quad (\lambda \text{ is a constant}). \tag{30}$$

So we obtain:

Conclusion 3. The special non-autonomous Birkhoff equations (25) of the rotational relativistic system possess a consistent algebraic structure according to the product (27).

From the product (27), we can define a new product

$$\widetilde{A}^* \circ \widetilde{B}^* \stackrel{\text{def}}{=} \widetilde{A}^* \widetilde{B}^* - \widetilde{B}^* \widetilde{A}^*, \tag{31}$$

which satisfies the Lie algebraic axiom

$$\widetilde{A}^* \circ \widetilde{B}^* + \widetilde{B}^* \circ \widetilde{A}^* = 0, \tag{32}$$

$$(\widetilde{A}^* \circ \widetilde{B}^*) \circ \widetilde{C}^* + (\widetilde{B}^* \circ \widetilde{C}^*) \circ \widetilde{A}^* + (\widetilde{C}^* \circ \widetilde{A}^*) \circ \widetilde{B}^* = 0. \tag{33}$$

So we obtain:

Conclusion 4. The special non-autonomous Birkhoff equations (25) of the rotational relativistic system possess a consistent Lie algebraic structure according to the product (27).

4. Poisson integrals of a rotational relativistic Birkhoff system

Because the autonomous and semi-autonomous Birkhoff equations (9) of a rotational relativistic system possess Lie algebraic structure, the Poisson theory integrating the holonomic conserved system can be used to the system. So we have:

Conclusion 5. $\widetilde{\Phi}^*(t, a^\mu) = \Phi^*(I_i(t, a^\mu), t, a^\mu) = C$ is a first integral of the autonomous and semi-autonomous Birkhoff equations (9) of a rotational relativistic system if and only if

$$\frac{\partial \widetilde{\Phi}^*}{\partial t} + [\widetilde{\Phi}^*, \widetilde{B}^*] = 0, \tag{34}$$

where $[\widetilde{\Phi}^*, \widetilde{B}^*] = \widetilde{\Phi}^* \circ \widetilde{B}^*$.

Proof $\widetilde{\Phi}^*$ is a first integral, then we have

$$\left.\frac{\mathrm{d}\widetilde{\Phi}^*}{\mathrm{d}t}\right|_{(10)} = \frac{\partial \Phi^*}{\partial t} + \sum_{i=1}^{N} \frac{\partial \Phi^*}{\partial I_i} \frac{\partial I_i}{\partial t} + \left(\frac{\partial \Phi^*}{\partial a^\mu} + \sum_{i=1}^{N} \frac{\partial \Phi^*}{\partial I_i} \frac{\partial I_i}{\partial a^\mu} \right) \widetilde{\Omega}^{*\mu\nu} \times \left(\frac{\partial B^*}{\partial a^\nu} + \sum_{i=1}^{N} \frac{\partial B^*}{\partial I_i} \frac{\partial I_i}{\partial a^\nu} \right) = 0,$$

noticing

$$\frac{\partial \widetilde{\Phi}^*}{\partial t} = \frac{\partial \Phi^*}{\partial t} + \sum_{i=1}^{N} \frac{\partial \Phi^*}{\partial I_i} \frac{\partial I_i}{\partial t},$$

$$\frac{\partial \widetilde{\Phi}^*}{\partial a^\mu} = \frac{\partial \Phi^*}{\partial a^\mu} + \sum_{i=1}^{N} \frac{\partial \Phi^*}{\partial I_i} \frac{\partial I_i}{\partial a^\mu},$$

$$\frac{\partial \widetilde{B}^*}{\partial a^\nu} = \frac{\partial B^*}{\partial a^\nu} + \sum_{i=1}^{N} \frac{\partial B^*}{\partial I_i} \frac{\partial I_i}{\partial a^\nu}. \tag{35}$$

So Eq.(34) holds and vice versa.

Equation (34) is the generalized Poisson conditions about the first integral of the autonomous and semi-autonomous rotational relativistic Birkhoff system.

For the autonomous system, the Birkhoffian $\widetilde{B}^*$ of the rotational relativistic system does not include time t, so $\dfrac{\partial \widetilde{B}^*}{\partial t} = 0$ holds. Substituting $\widetilde{\Phi}^* = \widetilde{B}^*$ into Eq.(34), it holds evidently. So we have:

Conclusion 6. The Birkhoffian of an autonomous rotational relativistic Birkhoff system is the first integral of the system.

If the rotational relativistic Birkhoff system (9) possesses the following two first integrals which are mutual absolute

$$\widetilde{\Phi}_1^* = \widetilde{\Phi}_1^*(t, a^\mu) = \Phi_1(I_i(t, a^\mu), t, a^\mu) = C_1,$$

$$\widetilde{\Phi}_2^* = \widetilde{\Phi}_2^*(t, a^\mu) = \Phi_2(I_i(t, a^\mu), t, a^\mu) = C_2, \tag{36}$$

then they must satisfy the generalized Poisson condition (34). So we have

$$\frac{\partial \widetilde{\Phi}_1^*}{\partial t} + [\widetilde{\Phi}_1^*, \widetilde{B}^*] = 0, \quad \frac{\partial \widetilde{\Phi}_2^*}{\partial t} + [\widetilde{\Phi}_2^*, \widetilde{B}^*] = 0. \tag{37}$$

Constructing a generalized Poisson bracket $[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*]$, and solving its partial derivative with respect to t, we obtain

$$\frac{\partial}{\partial t}[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*] = \left[\frac{\partial \widetilde{\Phi}_1^*}{\partial t}, \widetilde{\Phi}_2^*\right] + \left[\widetilde{\Phi}_1^*, \frac{\partial \widetilde{\Phi}_2^*}{\partial t}\right] + \frac{\partial \widetilde{\Phi}_1^*}{\partial a^\mu} \frac{\partial \widetilde{\Omega}^{*\mu\nu}}{\partial t} \frac{\partial \widetilde{\Phi}_2^*}{\partial a^\nu}. \tag{38}$$

Note that $\widetilde{R}_\nu^*$ of the autonomous and semi-autonomous rotational relativistic Birkhoff systems does not include time t, so we have $\dfrac{\partial \widetilde{R}_\nu^*}{\partial t} = 0$ and $\dfrac{\partial \widetilde{\Omega}^{*\mu\nu}}{\partial t} = 0$. Furthermore, we obtain

$$\frac{\partial}{\partial t}[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*] = \left[\frac{\partial \widetilde{\Phi}_1^*}{\partial t}, \widetilde{\Phi}_2^*\right] + \left[\widetilde{\Phi}_1^*, \frac{\partial \widetilde{\Phi}_2^*}{\partial t}\right]. \tag{39}$$

Using the Lie algebraic axioms (17) and (18), we have

$$\begin{aligned} &[[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*], \widetilde{B}^*] \\ =& -[[\widetilde{\Phi}_2^*, \widetilde{B}^*], \widetilde{\Phi}_1^*] - [[\widetilde{B}^*, \widetilde{\Phi}_1^*], \widetilde{\Phi}_2^*] \\ =& [\widetilde{\Phi}_1^*, [\widetilde{\Phi}_2^*, \widetilde{B}^*]] + [\widetilde{\Phi}_2^*, [\widetilde{B}^*, \widetilde{\Phi}_1^*]]. \end{aligned} \tag{40}$$

Considering Eqs.(39) and (40), and noting Eq.(37), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*] + [[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*], \widetilde{B}^*] \\ =& \left[\frac{\partial \widetilde{\Phi}_1^*}{\partial t} + [\widetilde{\Phi}_1^*, \widetilde{B}^*], \widetilde{\Phi}_2^*\right] \\ &+ \left[\widetilde{\Phi}_1^*, \frac{\partial \widetilde{\Phi}_2^*}{\partial t} + [\widetilde{\Phi}_2^*, \widetilde{B}^*]\right] = 0. \end{aligned} \tag{41}$$

In contrast with the generalized Poisson condition (34), we know $[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*]$ is an integral of the system. So we have:

Conclusion 7. If the autonomous and semi-autonomous rotational relativistic Birkhoff system (9) possesses two integrals $\widetilde{\Phi}_1^*(t, a^\mu) = C_1$ and $\widetilde{\Phi}_2^*(t, a^\mu) = C_2$ which are not mutual involution, then their generalized Poisson bracket $[\widetilde{\Phi}_1^*, \widetilde{\Phi}_2^*]$ is also a first integral of the system.

If $\widetilde{\Phi}^*(t, a^\mu) = C$ is a first integral including time t of the autonomous rotational relativistic Birkhoff system, solving the partial derivative of Eq.(34) with respect to t, and noting $\dfrac{\partial \widetilde{B}^*}{\partial t} = \dfrac{\partial \widetilde{\Omega}^{*\mu\nu}}{\partial t} = 0$, then we have

$$\frac{\partial}{\partial t}\left(\frac{\partial \widetilde{\Phi}^*}{\partial t}\right) + \left[\frac{\partial \widetilde{\Phi}^*}{\partial t}, \widetilde{B}^*\right] = 0. \tag{42}$$

Hence $\dfrac{\partial \widetilde{\Phi}^*}{\partial t}$ is a first integral of the system. So we have:

Conclusion 8. If the autonomous rotational relativistic Birkhoff system possesses a first integral $\widetilde{\Phi}^*(t, a^\mu) = C$ including time t, then $\dfrac{\partial \widetilde{\Phi}^*}{\partial t}, \dfrac{\partial^2 \widetilde{\Phi}^*}{\partial t^2}, \cdots$ are also first integrals of the system.

Similarly, we can prove the following conclusions.

Conclusion 9. If the autonomous and semi-autonomous rotational relativistic Birkhoff systems possess an integral including a^ρ, $\widetilde{\Omega}^{*\mu\nu}$ and $\widetilde{B}^*$ do not include a^ρ, then $\dfrac{\partial \widetilde{\Phi}^*}{\partial a^\rho}, \dfrac{\partial^2 \widetilde{\Phi}^*}{\partial a^{\rho 2}}, \cdots$ are also first integrals of the system.

Because the special non-autonomous rotational relativistic Birkhoff system (25) possesses a consistent Lie algebraic structure, but no Lie algebraic structure, the Poisson theory can be partly used to the system. So we have:

Conclusion 10. $\widetilde{\Phi}^*(t, a^\mu) = C$ is a first integral of the special non-autonomous rotational relativistic Birkhoff system (25), if and only if

$$\frac{\partial \widetilde{\Phi}^*}{\partial t} + \widetilde{\Phi}^* \widetilde{B}^* = 0. \tag{43}$$

Proof $\widetilde{\Phi}^*$ is a first integral, then we have

$$\begin{aligned}\left.\frac{\partial\widetilde{\Phi}^*}{\partial t}\right|_{(27)}=&\frac{\partial\Phi^*}{\partial t}+\sum_{i=1}^{N}\frac{\partial\Phi^*}{\partial I_i}\frac{\partial I_i}{\partial t}\\&+\left(\frac{\partial\Phi^*}{\partial a^\mu}+\sum_{i=1}^{N}\frac{\partial\Phi^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\widetilde{S}^{*\mu\nu}\\&\times\left(\frac{\partial B^*}{\partial a^\nu}+\sum_{i=1}^{N}\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}\right)\\=&\frac{\partial\widetilde{\Phi}^*}{\partial t}+\widetilde{\Phi}^*\widetilde{B}^*=0,\end{aligned}$$

and vice versa.

Equation (43) is the generalized Poisson condition about the first integral of the special non-autonomous rotational relativistic Birkhoff system (25).

Substituting $\Phi^*=B^*$ into Eq.(43), we obtain

$$\begin{aligned}&\frac{\partial B^*}{\partial t}+\sum_{i=1}^{N}\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial t}+\left(\frac{\partial B^*}{\partial a^\mu}+\sum_{i=1}^{N}\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\widetilde{S}^{*\mu\nu}\\&\times\left(\frac{\partial B^*}{\partial a^\nu}+\sum_{i=1}^{N}\frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}\right)\\=&\frac{\partial\widetilde{B}^*}{\partial t}+\frac{\partial\widetilde{B}^*}{\partial a^\mu}\widetilde{S}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\nu}\\=&\frac{\partial\widetilde{B}^*}{\partial t}+\frac{\partial\widetilde{B}^*}{\partial a^\mu}\widetilde{\Omega}^{*\mu\nu}\frac{\partial\widetilde{R}^*_\nu}{\partial t}=0.\end{aligned}\tag{44}$$

So we have:

Conclusion 11. $\widetilde{B}^*(t,a^\mu)=B^*(I_i(t,a^\mu),t,a^\mu)=C$ is the first integral of the special non-autonomous rotational relativistic Birkhoff system (25), if and only if Eq.(44) holds.

Solving the partial derivative of Eq.(43) with respect to a^ρ, we obtain

$$\begin{aligned}&\frac{\partial}{\partial a^\rho}\left(\frac{\partial\widetilde{\Phi}^*}{\partial t}\right)+\frac{\partial}{\partial a^\rho}\left(\frac{\partial\widetilde{\Phi}^*}{\partial a^\mu}\right)\widetilde{S}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\nu}\\&+\frac{\partial\widetilde{\Phi}^*}{\partial a^\mu}\frac{\partial}{\partial a^\rho}\left(\widetilde{S}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\nu}\right)\\=&\frac{\partial}{\partial t}\left(\frac{\partial\widetilde{\Phi}^*}{\partial a^\rho}\right)+\frac{\partial\widetilde{\Phi}^*}{\partial a^\rho}\widetilde{B}^*+\frac{\partial\widetilde{\Phi}^*}{\partial a^\mu}\frac{\partial}{\partial a^\rho}\left(\widetilde{S}^{*\mu\nu}\frac{\partial\widetilde{B}^*}{\partial a^\nu}\right)=0.\end{aligned}\tag{45}$$

So we have:

Conclusion 12. If the special non-autonomous rotational relativistic Birkhoff system (25) possesses an integral including a^ρ, $\widetilde{S}^{*\mu\nu}$ and $\widetilde{B}^*$ do not include a^ρ, then $\dfrac{\partial\widetilde{\Phi}^*}{\partial a^\rho}$, $\dfrac{\partial^2\widetilde{\Phi}^*}{\partial a^{\rho2}},\cdots$ are also first integrals of the system.

5. Example

We study the fixed-axis rotation of a mechanical system with a constant moment of force M_0. Its rotational relativistic Birkhoffian and Birkhoff functions are respectively

$$B^*=\frac{I}{M_0}\Gamma^2-a^1,\quad R_1^*=0,$$

$$R_2^*=\left(\frac{I}{M_0}\Gamma^2-a^1\right)\frac{I}{M_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{-1},\tag{46}$$

where $I=I_0\left(1-\dfrac{(a^2)^2}{\Gamma^2}\right)^{-1/2}$, and Γ is a limiting angular velocity. We attempt to study the system's integrals by using Poisson theory.

We can obtain the Birkhoff contravariant tensors of the second-order rotational relativistic Birkhoff system from R_μ^*

$$(\Omega^{*\mu\nu})=\begin{pmatrix}0&\dfrac{M_0}{I}\left(1-\dfrac{(a^2)^2}{\Gamma^2}\right)\\-\dfrac{M_0}{I}\left(1-\dfrac{(a^2)^2}{\Gamma^2}\right)&0\end{pmatrix}.\tag{47}$$

The system is a second-order autonomous rotational relativistic Birkhoff system. Using conclusion 6, we know that the system has an integral

$$\Phi^*=B^*=\frac{I}{M_0}\Gamma^2-a^1=C_1.\tag{48}$$

Using conclusion 5, it is easily proven that

$$\Phi_1^*=\frac{I}{M_0}\Gamma^2-2a^1+a^2t=C_1.\tag{49}$$

Using conclusion 8, we know that the autonomous rotational relativistic Birkhoff system possesses a first integral Φ_1^* including time t, and we can construct a new integral from Φ_1^*

$$\Phi_2^*=\frac{\partial\Phi_1^*}{\partial t}=a^2=C_2.\tag{50}$$

Constructing a generalized Poisson bracket $[\Phi_1^*,\Phi_2^*]$, we obtain

$$\begin{aligned}[\Phi_1^*,\Phi_2^*]=&\frac{\partial\Phi_1^*}{\partial a^\mu}\Omega^{*\mu\nu}\frac{\partial\Phi_2^*}{\partial a^\nu}\\=&-2\frac{M_0}{I}\left(1-\frac{(a^2)^2}{\Gamma^2}\right).\end{aligned}$$

Using conclusion 7, we know that

$$\Phi_3^*=[\Phi_1^*,\Phi_2^*]=-\frac{2M_0}{I}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)=C_3\tag{51}$$

is also a first integral of the system.

6. Discussion

We have proven that the canonical equations of a rotational relativistic Hamilton system are a special example of Eqs.(4) in Ref.[27]. In this paper, if R^*_μ does not include time t, for the rotational relativistic Birkhoff system (4), we let

$$a^\mu = \begin{cases} q^\mu & (\mu = 1, \cdots, n) \\ p_{\mu-n} & (\mu = n+1, \cdots, 2n); \end{cases}$$

$$R^*_\mu = \begin{cases} p_\mu & (\mu = 1, \cdots, n), \\ 0 & (\mu = n+1, \cdots, 2n), \end{cases}$$

$$B^* = H_r, \tag{52}$$

then the Birkhoff tensors of the rotational relativistic system are

$$(\widetilde{\Omega}^*_{\mu\nu}) = \begin{bmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{bmatrix}. \tag{53}$$

We can obtain immediately the canonical equations of the rotational relativistic Hamilton system

$$\dot{q}^s = \frac{\partial H_r}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H_r}{\partial q^s} \quad (s = 1, \cdots, n), \tag{54}$$

where

$$H_r(t, q^s, p_s) = p_s \dot{q}^s - L_r, \quad p_s = \frac{\partial L_r}{\partial \dot{q}_s},$$

$$L_r(t, q^s, \dot{q}^s) = T^*_r - V,$$

$$T^*_r = I_{0i} \Gamma_i^2 (1 - \sqrt{1 - \dot{\theta}_i^2 / \Gamma_i^2})$$

$$(s = 1, \cdots, n; i = 1, \cdots, N). \tag{55}$$

Therefore, in this paper we give the algebraic structures and Poisson integrals[8] of a rotational relativistic Hamilton system.

Under the classical approximation $\dot{\theta}_i \ll \Gamma_i$

$$I_i = \frac{I_{0i}}{\sqrt{1 - \dot{\theta}_i^2 / \Gamma_i^2}} \approx I_{0i} \quad (i = 1, \cdots, N), \tag{56}$$

$$\widetilde{B}^* = B^*(I_i(t, a^\mu), t, a^\mu) \approx B(t, a^\mu),$$

$$\widetilde{R}^*_\nu = R^*_\nu(I_i(t, a^\mu), t, a^\mu) \approx R_\nu(t, a^\mu). \tag{57}$$

Therefore, in this paper we give the algebraic structures and Poisson integrals of a classical rotational Birkhoff system.

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