Vol 11 No 5, May 2002
1009-1963/2002/11(05)/0486-06

Chinese Physics

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k-quantum nonlinear Jaynes–Cummings model in two trapped ions*

Zeng Hao-Sheng(曾浩生)[a)b)], Kuang Le-Man(匡乐满)[b)], and Gao Ke-Lin(高克林)[a)]

[a)] *State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan* 430071, *China*

[b)] *Department of Physics, Hunan Normal University, Changsha* 410081, *China*

(Received 12 September 2001; revised manuscript received 27 November 2001)

We have obtained a k-quantum nonlinear Jaynes–Cummings model for two trapped ions interacting with laser beams resonant to the kth red side-band of the centre-of-mass mode, far from the Lamb–Dicke regime. The exact analytic solution shows the existence of quantum collapses and revivals of the occupation of two atoms.

Keywords: nonlinear Jaynes–Cummings model, occupation, collapse and revival
PACC: 4250, 3280

Since Jaynes and Cummings first put forward what has become the Jaynes–Cummings model (JCM),[1] there has arisen a sizeable scientific industry devoted to exploiting and extending this work, because of its interesting nonclassical effects, such as collapses and revivals of Rabi oscillations,[2] squeezing[3] and antibunched effects.[4] The extensions of the JCM have been mainly in two directions. One is to extend it to multi-photon processes[5] and multi-atom processes[6] where a single-mode quantized field and two-level atom or atoms are still involved. The other is to generalize it to situations that include multi-level atoms and multi-mode quantized fields.[7] The experimental realization of the JCM with dissipation has been demonstrated both in microwave and in optical domain. The observation of quantum revivals and nonclassical photon statistics with Rydberg atoms and microwave cavities has been reported,[8] and in the optical regime vacuum Rabi splitting has been observed.[9]

Recently, another system to exhibit JCM dynamics, e.g. a trapped and laser-irradiated ion, has been proposed. In this system, the quantized centre-of-mass motion of the ion in the trap potential plays the role of the boson mode, which is coupled via the laser to the internal degrees of freedom. An analogy between an undamped trapped ion and the JCM has been noted by Blockley *et al.*[10] and by Cirac *et al.*[11] for damped trapped ions. A study of nonlinear multi-quantum JCM dynamics of a trapped ion is presented by Vogel and de Matos Filho.[12] Furthermore, a trapped ion in any position of a standing wave also exhibits JCM dynamics.[13] A significant feature of the trapped-ion JCM is that the effective coupling constant, which depends on Rabi frequency, could be readily adjusted in experiments to satisfy the strong coupling condition.

More recently, much attention has been focused on studying multiple trapped ions. Theoretical methods to generate the electronic maximal entangled state[14] and to generate nonclassical vibratile states of multiple ions[15] have been put forward. For two trapped ions, a theoretical scheme for the generation of Bell electronic states out of the Lamb–Dicke regime has been proposed, and some applications have been discussed.[16] In experiments, entangled states up to four ions have been realized, and some applications, such as the proof of Bell's inequality and decoherence-free quantum memory, have been confirmed.[17] These improvements provide some advantageous conditions for studying properties of the multi-ion JCM.

In this paper we demonstrate that a system comprising two trapped ions interacting with a laser beam[14] resonant to the kth red side-band of the centre-of-mass mode, far from the Lamb–Dicke regime, shows a nonlinear JCM dynamics. Collapses and revivals can be observed in this system. These properties may be used to realize the quantum non-demolition measurement of the final temperatures or the vibrational energy of the trapped ions by applying a sequence of probe pulses or a CW probe field.[18]

Consider two ions trapped in a linear radio-

*Project supported by the National Natural Science Foundation of China (Grant Nos 19734006 and 10075018).

frequency trap which are strongly bound in the y- and z-directions but weakly bound in a harmonic potential in the x-direction. Due to the Coulomb force between them, their motion couples together and forms the centre-of-mass mode and breathing mode. A more detailed description of the concept of cold-trapped ion quantum computation may be consulted in Ref.[19]. We assume that the two ions are illuminated simultaneously with a dispersive beam resonant to the kth red side-band of the centre-of-mass mode, so that the effects from the breathing mode can be neglected because of the very large off-resonance.[14] The Hamiltonian for this system is

$$H = H_0 + H_{\rm int}, \tag{1}$$

$$H_0 = \nu(a^+a + 1/2) + \omega\sum_{i=1}^{2}\sigma_{iz}/2, \tag{2}$$

$$H_{\rm int} = \sum_{i=1}^{2}\frac{\Omega}{2}\left\{\sigma_{+i}{\rm e}^{{\rm i}\eta(a+a^+)}{\rm e}^{-{\rm i}(\omega-k\nu)t} + {\rm H.c.}\right\}, \tag{3}$$

where ν and a^+ (a) are the frequency and ladder operators of the centre-of-mass mode, and ω is the energy difference between the ground state $|0\rangle$ and the long-lived metastable excited state $|1\rangle$ of each ion. For simplicity we further assume that the Rabi frequency and the Lamb–Dicke parameter for the two ions are the same. Transforming the Hamiltonian (3) to the interaction picture with respect to H_0 and making use of the rotating wave approximation, we have

$$H_{\rm int} = \Omega J_+ f(a^+a)a^k + {\rm H.c.} \tag{4}$$

with

$$f(a^+a) = {\rm e}^{-\eta^2/2}\sum_{n=0}^{\infty}\frac{({\rm i}\eta)^{2n+k}}{n!(n+k)!}(a^+)^n a^n, \tag{5}$$

where $J_\pm = J_x \pm {\rm i}J_y$ with $J_\alpha = \frac{1}{2}(\sigma_{1\alpha} + \sigma_{2\alpha})$ ($\alpha = x,\ y,\ z$) being the three components of total spin operator of the two ions.

Equation (4) is the so-called k-quantum nonlinear JCM Hamiltonian. When any one of the two ions is excited, the quantum number of the J_z component of the total spin operator increases by one and, in the meantime, the quantum number of the centre-of-mass mode decreases by k.

Noting that, in this system, the total number of excitations (including electron excitations and excitations of the centre-of-mass mode) is conserved, we can arrange the eigenstates of Hamiltonian H_0 in the order of total excitations,

$$|1,\ n-2k\rangle^n,\ |0,\ n-k\rangle^n,\ |-1,\ n\rangle^n;$$
$$|1,\ n-k\rangle^{n+k},\ |0,\ n\rangle^{n+k},\quad |-1,\ n+k\rangle^{n+k};\cdots,$$

where the first number in each Dirac symbol represents the quantum number of the J_z component and the second represents the phonon number of the centre-of-mass mode. The superscript above each Dirac symbol represents the total excitations (note that it must deplete k phonons for exciting one atom, so an atom being in the excited state includes k excitations).

In the Hilbert space constituted on the above basis, the interaction Hamiltonian (4) is a diagonal matrix made up of 3×3 sub-matrices. Every sub-matrix represents a subspace corresponding to a definite number of excitations. In the subspace including n excitations: $|1,\ n-2k\rangle^n$, $|0,\ n-k\rangle^n$, $|-1,\ n\rangle^n$, the Hamiltonian (4) can be written as

$$H_{\rm int} = \begin{bmatrix} 0 & \Omega^{(n-2k)(n-k)} & 0 \\ (-1)^k\Omega^{(n-2k)(n-k)} & 0 & \Omega^{(n-k)n} \\ 0 & (-1)\Omega^{(n-k)n} & 0 \end{bmatrix}, \tag{6}$$

where

$$\Omega^{l(l+k)} = \sqrt{2}\Omega\langle l|\,{\rm e}^{{\rm i}\eta(a+a^+)}\,|l+k\rangle = \sqrt{2}\Omega{\rm e}^{-\eta^2/2}({\rm i}\eta)^k\sqrt{\frac{l!}{(l+k)!}}L_l^k(\eta^2) \tag{7}$$

is the kth Rabi frequency and

$$L_n^k(x) = \sum_{m=0}^{n}(-1)^m\begin{pmatrix} n+k \\ n-m\end{pmatrix}\frac{x^m}{m!} \tag{8}$$

is the generalized Laguerre polynomial. The exact propagator for this Hamiltonian then reads as

$$U = {\rm e}^{-{\rm i}H_{\rm int}t} = \begin{bmatrix} U_{11} & U_{10} & U_{1-1} \\ U_{01} & U_{00} & U_{0-1} \\ U_{-11} & U_{-10} & U_{-1-1}\end{bmatrix} \tag{9}$$

with

$$U_{11} = \frac{1}{A_k^2+B_k^2}\left[A_k^2\cos\left(\sqrt{A_k^2+B_k^2}t'\right) + B_k^2\right], \tag{10}$$

$$U_{00} = \cos\left(\sqrt{A_k^2 + B_k^2}t'\right), \tag{11}$$

$$U_{-1-1} = \frac{1}{A_k^2 + B_k^2}\left[B_k^2 \cos\left(\sqrt{A_k^2 + B_k^2}t'\right) + A_k^2\right], \tag{12}$$

$$U_{10} = -U_{01}^* = \frac{-\mathrm{i}^{k+1}A_k}{\sqrt{A_k^2 + B_k^2}} \sin\left(\sqrt{A_k^2 + B_k^2}t'\right), \tag{13}$$

$$\begin{aligned} U_{1-1} =& U_{-11} \\ =& (-1)^{k+1}\frac{A_k B_k}{A_k^2 + B_k^2} \\ & \times \left[1 - \cos\left(\sqrt{A_k^2 + B_k^2}t'\right)\right], \end{aligned} \tag{14}$$

$$\begin{aligned} U_{0-1} =& -U_{-10}^* \\ =& \frac{-\mathrm{i}^{k+1}B_k}{\sqrt{A_k^2 + B_k^2}} \sin\left(\sqrt{A_k^2 + B_k^2}t'\right), \end{aligned} \tag{15}$$

where $t' = \Omega t$ and

$$\begin{aligned} A_k =& A(n,k,\eta) \\ =& \sqrt{2}\mathrm{e}^{-\eta^2/2}\eta^k \sqrt{\frac{(n-2k)!}{(n-k)!}} L_{(n-2k)}^k\left(\eta^2\right), \end{aligned} \tag{16}$$

$$\begin{aligned} B_k =& B(n,k,\eta) \\ =& \sqrt{2}\mathrm{e}^{-\eta^2/2}\eta^k \sqrt{\frac{(n-k)!}{n!}} L_{(n-k)}^k\left(\eta^2\right). \end{aligned} \tag{17}$$

The subscripts $1, 0, -1$ in each matrix element U refer to the quantum number of the J_z component. (Noting that A_k and B_k are functions of variables n, k and η, we should set $k = 1$ for one-quantum resonance calculations and $k = 2$ for two-quantum resonance.)

In order to study the phenomena of collapses and revivals of the two-ion system, imitating the traditional process, we assume that initially the internal states of the two ions are both in the ground states and the external motional state of the centre-of-mass mode in a coherent state $|\alpha\rangle$. Thus the whole initial state of the system is

$$\rho(0) = \sum_{m',m''=0}^{\infty} q(m')q^*(m'')|-1\ m'\rangle\langle -1\ m''|, \tag{18}$$

where $q(m) = \exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^m}{\sqrt{m!}}$ are the coefficients for the coherent state expanding in the number state basis.

The involved density operator of the internal state of this system can be obtained by tracing over the vibrational motion of the centre-of-mass mode: $\rho_e(t) = \mathrm{Tr}_{\mathrm{vib}}(U\rho(0)U^+)$. With the above equations, we can obtain the time evolution of the diagonal elements of the reduced density matrix, which refers to the probabilities that the two ions are both in excited states (P_{11}), one in the excited state and the other in the ground state (P_{00}), and both in ground states (P_{-1-1}), respectively; for example, $P_{ii} = \rho_{ii} = \langle i|\,\rho_e(t)\,|i\rangle$. For one-quantum resonance, it reads

$$\begin{aligned} P_{11}(t) =& \sum_{n=2}^{\infty} \frac{A_1^2 B_1^2}{(A_1^2 + B_1^2)^2} \\ & \times \left[1 - \cos\left(\sqrt{A_1^2 + B_1^2}t'\right)\right]^2 p\,(n), \end{aligned} \tag{19}$$

$$\begin{aligned} P_{00}(t) =& \sum_{n=2}^{\infty} \frac{B_1^2}{A_1^2 + B_1^2}\left[\sin\left(\sqrt{A_1^2 + B_1^2}t'\right)\right]^2 p(n) \\ & + \left[\sin(Ct')\right]^2 p\,(1), \end{aligned} \tag{20}$$

$$\begin{aligned} P_{-1-1}(t) =& \sum_{n=2}^{\infty} \frac{1}{(A_1^2 + B_1^2)^2} \\ & \times \left[A_1^2 + B_1^2 \cos\left(\sqrt{A_1^2 + B_1^2}t'\right)\right]^2 p(n) \\ & + \left[\cos(Ct')\right]^2 p\,(1) + p\,(0) \end{aligned} \tag{21}$$

with $C = \sqrt{2}\mathrm{e}^{-\eta^2/2}\eta$, and

$$p(n) = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \tag{22}$$

is the phonon number distribution for the initial coherent state of the centre-of-mass mode, and $A_1 = A(n,1,\eta)$ and $B_1 = B(n,1,\eta)$ are defined by Eqs.(16) and (17).

Zhang *et al.*[20] have studied the dynamics of JCM for two atoms interacting with a single electromagnetic field for the M-photon process, taking into account dipole–dipole interaction of the two atoms. It is demonstrated that there are three series of collapses and revivals for the inversion occupation. Now for two trapped ions, when the internal states are nonlinearly coupled with the centre-of-mass mode via lasers, we find that there are two series of collapses and revivals for occupation probabilities of Eqs.(19)–(21). Below we study in detail the time evolution of occupation for different Lamb–Dicke parameters.

Firstly we consider the case of a small Lamb–Dicke parameter, e.g. in Lamb–Dicke limits. We set $\eta = 0.05$ and the initial average vibron number for the centre-of-mass mode $|\alpha|^2 = 15$. The occupation of different coupling internal levels is plotted in Fig.1. From these curves, we can see clearly the collapses and revivals of the occupation of all coupling internal levels

of the two trapped ions. Moreover, the effects of collapses and revivals become more obvious with the increase of initial average vibron number. These results are in agreement with the results of the standard JCM. In addition, the time evolution of $P_{00}(t)$ is similar to the occupation inversion of one trapped ion interacting with lasers for small Lamb-Dicke parameter.[10] But the time evolutions of occupation $P_{11}(t)$ and $P_{-1-1}(t)$ are almost inverse in phase. This is because with the parameters in Fig.1 (e.g. large average vibrons and small η), we have $A_1 \approx B_1$ and the last two terms in Eq.(21) may be left out. Thus we obtain $P_{11}(t) = \sum_{n=2}^{\infty} \left[\sin\left(\sqrt{A_1^2 + B_1^2}t'/2\right)\right]^4 p(n)$ and $P_{-1-1}(t) = \sum_{n=2}^{\infty} \left[\cos\left(\sqrt{A_1^2 + B_1^2}t'/2\right)\right]^4 p(n)$, e.g. $P_{11}(t)$ and $P_{-1-1}(t)$ are out of phase.

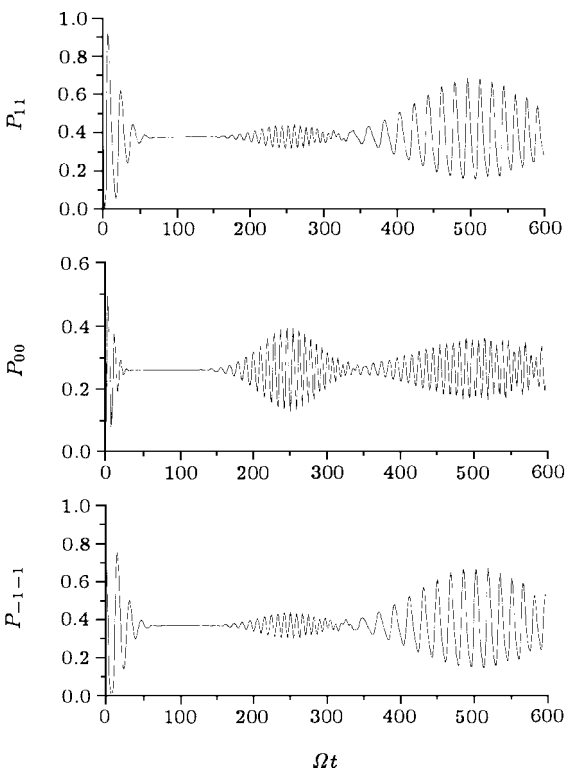


Fig.1. The occupations $P_{11}(t)$, $P_{00}(t)$, $P_{-1-1}(t)$ of electronic states of the two ions for one-quantum resonance as functions of dimensionless time with parameters $\eta = 0.05$, $|\alpha|^2 = 15$.

In fact, from the mathematical form of collapses and revivals in the standard JCM[1]

$$w(t) = -\frac{1}{2}\sum_{n=0}^{\infty} p(n)\cos(2\lambda t\sqrt{n}), \qquad (23)$$

we can see that there are two important factors for collapses and revivals. One is the Poisson distribution of the photon number $p(n)$, the other is the frequency of the cosine function satisfying $\omega \propto \sqrt{n}$, e.g. ω^2 increases linearly or approximately linearly with photon number n. In our problem, we find by calculation that, in the range of $|\alpha|^2$=10–40 and when $\eta < 0.3$, we have $A_k/B_k \approx 1$ either for one-quantum or for two-quantum resonance. Thus the weight for each series of collapse and revival closes on Poisson distribution. For the second problem, we have $\omega^2 = A_k^2 + B_k^2$ which is a function of n, k and η. For one-phonon resonance, Fig.2 shows the change of ω^2 with vibron number n for different Lamb–Dicke parameters. We see that there is almost a linearly increasing relation between ω^2 and n when $\eta = 0.05$. Therefore it is a case of good collapses and revivals in this case as in the standard JCM.

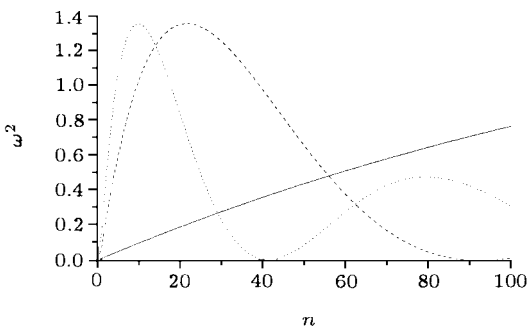


Fig.2. ω^2/Ω as a function of vibron number n for one-quantum resonance: solid line, $\eta = 0.05$; dashed line, $\eta = 0.2$; dotted line, $\eta = 0.3$.

When the Lamb–Dicke parameters become larger, the relation between ω^2 and n becomes complex (see Fig.2). The rule of collapse and revival also becomes intricate. In general, collapses and revivals in these cases are obscure. But for some combinations of η and n, it may become very explicit. In fact, under the condition of Poisson distribution of vibrons, the linear increasing relation between ω^2 and n may relax at the expense of the effect of collapses and revivals. As long as it is within the main range of Poisson distribution of vibrons, it is approximately linear. Furthermore, owing to some symmetry of Poisson distribution, an approximately linear decreasing form for ω^2–n will also display collapse or revival. This is proved by Fig.3 and later discussions. Combining Figs.2 and 3, we can see that for $\eta = 0.3$ and in the decreasing stage of the ω^2–n curve, visible collapse and revival exists. At the decreasing stage of the $\eta = 0.2$ curve (say $|\alpha|^2 = 40$), there is also clear collapse and revival, but these large average vibrons may go beyond the validity of the rotating wave approximation of Eq.(4), so we do not discuss it here. However, at the increasing stage of these two curves, collapse and revival becomes very obscure because of the very small average vibrons.

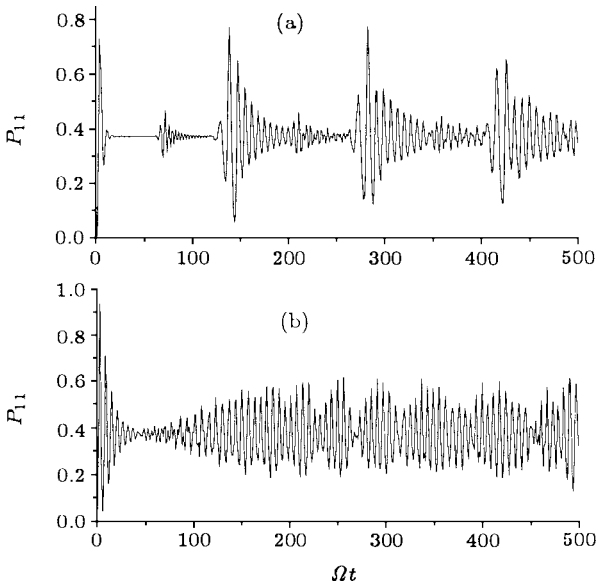


Fig.3. Time evolution of the occupation $P_{11}(t)$ of the excited electronic state of the two ions for one-quantum resonance versus the dimensionless time: (a) $\eta = 0.3$, $|\alpha|^2 = 25$; (b) $\eta = 0.2$, $|\alpha|^2 = 10$.

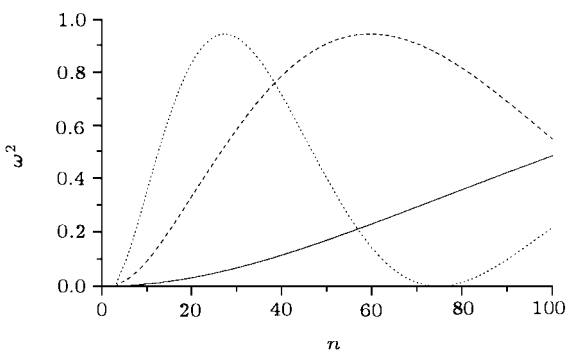


Fig.4. ω^2/Ω as a function of vibron number n for two-quantum resonance: solid line $\eta = 0.1$; dashed line $\eta = 0.2$; dotted line $\eta = 0.3$.

In order to clarify the collapses and revivals of two-quantum resonance, in view of the above discussions, we first plot the curves of ω^2–n for different Lamb–Dicke parameters in Fig.4. Conjecture and calculation suggest that for $\eta = 0.2$, in the increasing stage of the curve (about $|\alpha|^2$=10–40), there are visible collapses and revivals, and for $\eta = 0.3$, a round $|\alpha|^2 = 15$, there is a good collapse at the beginning of evolution (Fig.5(c)). In addition, from Figs.5(a) and 5(b) we see that the revival time becomes longer with the increase of average vibrons. This is also consistent with the result of the standard JCM.

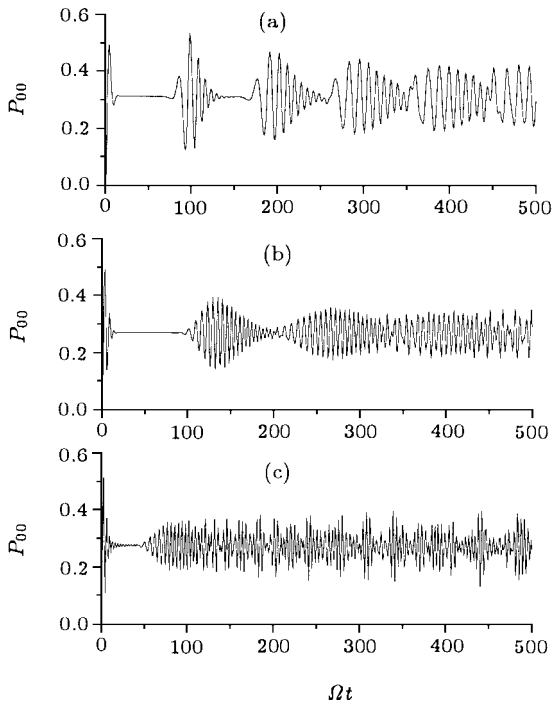


Fig.5. The occupation $P_{00}(t)$ of the coupling excited state of the two ions for two-quantum resonance as a function of dimensionless time with parameters: (a) $\eta = 0.2$, $|\alpha|^2 = 10$; (b) $\eta = 0.2$, $|\alpha|^2 = 20$; (c) $\eta = 0.3$, $|\alpha|^2 = 15$.

In conclusion, we have shown that the behaviour of two trapped ions, interacting with a laser beam resonant to the kth red side-band of the centre-of-mass mode, can be described by a nonlinear JCM within appropriate limits. An exact analytic solution for this type of JCM has been presented and the results indicated obvious collapses and revivals. The origin of collapses and revivals was analysed mathematically. We expect similar behaviours for more than two ions, and also expect new features of the nonclassical statistics of the vibrational mode and of other effects known from the standard JCM.

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