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New approach to the normal mode method in underwater acoustics*

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A new approach to the numerical solution of normal mode problems in underwater acoustics is presented, in which the corresponding normal mode problem is transformed to the problem of solving a dynamic system. Three applications are considered: (1) the broad band normal mode problem; (2) the range-dependent problem with perturbation proportional to the range parameter; and (3) the evolution of the normal mode with environmental parameters. A numerical simulation for a broad band problem is performed, and the calculated eigenvalues have good agreement with those obtained by the standard normal mode code KRAKAN.

Keywords: normal mode problem, dynamic system, numerical simulation
PACC: 0427, 4330

1. Introduction

Normal mode methods have been used widely in underwater acoustics. One of the earliest papers was published in 1948 by Pekeris.[1] Since then much progress has been made in the development of normal mode methods, and there are many models available based on normal mode methods. Stickler[2] introduced the famous branch line model to take into account the contributions of both the normal mode and branch line. One of the most widely used numerical codes for the normal mode problem in underwater acoustics is the KRAKAN code developed by Porter.[3,4] The fastest numerical code of the normal mode available today may be the WKBZ code proposed by Zhang.[5,6] Compared with Pekeris's original version, these later works allow for a more detailed description of environmental sound speed profile.

The purpose of this paper is to provide a new approach to the normal mode method which may be used to solve broad band and a certain type of range-dependent normal mode problems. The approach used here was originally used in quantum physics to investigate the spectrum evolution with respect to the strength parameter of classical nonintegrable perturbation. The main idea of the method is to transform an eigenvalue problem into a dynamic system problem. One of the merits of the approach is that it solves the broad band and range-dependent problems, and the other is that the method can be used to investigate the evolution of the eigenvalue and eigenfunction under a complex ocean environment. We show only the idea of the method in this paper; a detailed comparison with currently used codes will be reported separately elsewhere.

2. Generalized Calogero–Moser system and eigenvalue problem

The multiple avoided crossing or repulsion of energy levels is an essential mechanism for complicated spectra with a definite symmetry on quantum bound systems which are classically nonintegrable and chaotic.[7] In the 1980s, a new approach was proposed by Pechukas,[8] Yukawa,[9] and Nakamura and Lakshmanan[10] independently to investigate the spectrum evolution with respect to the strength parameter of classical nonintegrable perturbation. In this section, we give a brief sketch of their idea.

Let $H = H_0 + tV$ be the Hamiltonian for a quantum bound system whose classical version is nonintegrable. H_0 and V are a classically integrable part and a nonintegrable perturbation, respectively. By taking the nonintegrability parameter t as "time" variable, for the eigenvalue equation

$$H(t)|n(t)\rangle = x_n(t)|n(t)\rangle \tag{1}$$

one has the dynamic equations of motion for both

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eigenvalue and eigenfunction given by

$$\frac{\mathrm{d}}{\mathrm{d}t}x_n(t) = p_n(t) \equiv \langle n(t)|V|n(t)\rangle, \tag{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}p_n(t) = 2\sum_{m(\neq n)} V_{nm}V_{mn}(x_n - x_m)^{-1}, \tag{2b}$$

$$\frac{\mathrm{d}|n(t)\rangle}{\mathrm{d}t} = \sum_{m(\neq n)} |m(t)\rangle V_{mn}(x_n - x_m)^{-1}, \tag{2c}$$

$$\frac{\mathrm{d}\langle n(t)|}{\mathrm{d}t} = \sum_{m(\neq n)} \langle m(t)|V_{mn}(x_n - x_m)^{-1}, \tag{2d}$$

where the symbols $\langle\ |$ and $|\ \rangle$ denote the Dirac bra and ket vectors, respectively, $V_{mn} \equiv \langle m|V|n\rangle$. We define

$$\Lambda_{nm} = \mathrm{i}^{-1}V_{nm}(x_n - x_m), \tag{3}$$

which can be rewritten as

$$\Lambda_{\alpha\beta} = \langle\alpha|L_0^+ L_0|\beta\rangle + \lambda_0\delta_{\alpha\beta}, \tag{4}$$

where λ_0 is the lowest eigenvalue of Λ, and L is an appropriate time independent operator with unique inverse ${L_0}^{-1}$. Using the Λ defined in Eq.(3), Eqs.(2) become

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = 2\sum_{m(\neq n)} \Lambda_{nm}\Lambda_{mn}(x_n - x_m)^{-3}, \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{i}\Lambda_{nm}) = \sum_{l(\neq m,n)} \Lambda_{nl}\Lambda_{lm}[(x_m - x_l)^{-2} - (x_n - x_l)^{-2}]. \tag{6}$$

Equation (6), together with Eqs.(2a) and (5), is a generalized Calogero–Moser system in 1+1 dimensions with internal vector space. The Calogero–Moser system is a completely integrable system whose Hamiltonian structure has been constructed by Nakamura and Lakshmanan.[10] It is especially important to applications that the system can be solved in a completely algebraic way.

Let us define matrices P, X, Γ and L as

$$\begin{aligned} P_{nm} &= \delta_{nm}p_n + (1-\delta_{nm})\mathrm{i}\Lambda_{nm}(x_n - x_m)^{-1}, \\ X_{nm} &= \delta_{nm}x_n, \\ \Gamma_{nm} &= (1-\delta_{nm})\mathrm{i}\Lambda_{nm}(x_n - x_m)^{-2}, \\ L &\equiv \{L_0|1\rangle, L_0|2\rangle, \cdots, L_0|n\rangle, \cdots, L_0|N\rangle\}, \end{aligned} \tag{7}$$

where $\{L_0|\ \rangle\}$ are N-component column vectors. A Lax representation of Eq.(7) is then written as

$$\begin{aligned} &\frac{\mathrm{d}P}{\mathrm{d}t} = [\Gamma, P], \quad \frac{\mathrm{d}X}{\mathrm{d}t} = [\Gamma, X] + P, \\ &\frac{\mathrm{d}L}{\mathrm{d}t} = -L\Gamma, \quad \frac{\mathrm{d}L^+}{\mathrm{d}t} = \Gamma L^+. \end{aligned} \tag{8}$$

These matrix equations can be solved in a completely algebraic way:

$$\begin{aligned} &X(t) = U(t)[X(0) + P(0)t]U^{-1}(t), \\ &P(t) = U(t)P(0)U^{-1}(t), \\ &L(t) = L(0)U^{-1}(t), \quad L^+(t) = U(t)L^+(0), \end{aligned} \tag{9}$$

and the matrices $U(t)$ and $X(t)$ can be easily found by diagonalizing $[X(0) + P(0)t]$.[11]

3. Application to normal mode theory

The approach presented above is a universal one which can be used for any eigenvalue problem of the type of Eq.(1). In this section, we show how to apply the method to ocean acoustic problems. For a 2D shallow water sound propagation problem, the governing equation is (a harmonic time dependence $\exp(-\mathrm{i}\omega t)$ being assumed)

$$\begin{aligned} &\rho(x,z)\left[\frac{\partial}{\partial x}\frac{1}{\rho(x,z)}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\frac{1}{\rho(x,z)}\frac{\partial}{\partial z}\right]p(x,z) \\ &+ \frac{\omega^2}{c(x,z)^2}p(x,z) = 0, \end{aligned} \tag{10}$$

where $p(\omega; x, z)$ is the acoustic pressure, and $c(x,z)$ and $\rho(x,z)$ are, respectively, the sound speed and density of the medium as functions of depth z and range x.

Assuming $\mathrm{d}\rho(x,z)/\mathrm{d}x = 0$, i.e. the density is range independent, and treating x as a parameter, the local modal function in the range x satisfies the following modal equation

$$\begin{aligned} &\rho(z)\left[\frac{\partial}{\partial z}\frac{1}{\rho(z)}\frac{\partial}{\partial z}\right]\varphi(x,z) + \frac{\omega^2}{c(z,x)^2}\varphi(x,z) \\ &= k^2\varphi(x,z), \end{aligned} \tag{11}$$

with appropriate boundary conditions.[3] In Eq.(11), $\varphi(x,z)$ is the depth modal function of normal mode at the range x, k is the horizontal wavenumber. For a fixed range x, the modal equation is a classical Sturm Liouville eigenvalue problem. This is a basic problem in underwater acoustics to find the normal modal solution. We show that the algorithm given in section 2 can be used to solve the broad band eigenvalue problem of range-independent problems, and a certain type of range-dependent problem, and the evolution of modes with the environmental parameters.

3.1. Broad band eigenvalue problem

For the range-independent problem, Eq.(11) is reduced to

$$\rho_0(z)\left[\frac{\partial}{\partial z}\frac{1}{\rho_0(z)}\frac{\partial}{\partial z}\right]\varphi(z)+\frac{\omega^2}{c_0(z)^2}\varphi(z)=k^2\varphi(z). \quad (12)$$

Using any conventional normal mode code, one can solve numerically the problem for a given frequency ω. However, to calculate the broad band problem one must cost much computational time in using the code repeatedly. Here we shall apply the algorithm given in section 2 to solve the broad band problem and together with the KARKAN code for comparison.

We define

$$H_0 \equiv \rho_0\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\rho_0}\frac{\mathrm{d}}{\mathrm{d}z}\right)\right]+\frac{\omega_0^2}{c(z)^2}, \quad (13)$$

$$H(\omega) \equiv H_0+\frac{\omega^2-\omega_0^2}{c(z)^2}. \quad (14)$$

The eigenvalue problem (13) and (14) has been transformed to the same form as Eq.(1). The difference between the two problems is that the system in section 2 is usually defined for bound systems, while the system here is in principle an infinite system. The corresponding generalized Calogero–Moser system is also an infinite system. To obtain a bound system we truncate the infinite system at the finite order N. The standard way to transform an infinite-dimensional problem into a finite-dimensional problem is to project the infinite system into the space spanned by finite orthogonal functions base. For the present problem, for given frequency ω_0, we have the corresponding bound system defined by

$$[H(\omega)]_{mn}=(\varphi_n, H(\omega)\varphi_m),$$
$$[V]_{mn}=(\varphi_m, V\varphi_n), \quad n,m=1,2,\cdots,N, \quad (15)$$

i.e.

$$[H(\omega)]_{mn}=[H(\omega_0)]_{mn}+(\omega^2-\omega_0^2)[V]_{mn}, \quad (16)$$

$$x_n(\omega)\equiv k_n^2(\omega), \quad x_n(\omega_0)\equiv k_n^2(\omega_0). \quad (17)$$

The orthogonal functions $\varphi_n(n=1,2,\cdots,N)$ are eigenfunctions for the differential operator H_0 which are calculated by the KRAKAN code in our numerical example. In this way, we have projected the infinite-dimensional system into a finite system. For this finite system, N-component vectors $|n\rangle(\omega)(n=1,2,\cdots,N)$ in fact are expanding coefficient vectors with respect to the functions $\varphi_n(n=1,2,\cdots,N)$. By treating $t=(\omega^2-\omega_0^2)$ as a "time" variable, a broad band problem can be solved by diagonalizing $[X(0)+P(0)(\omega^2-\omega_0^2)]$ as defined by Eq.(9). A numerical example is given for the broad band problem in section 4.

3.2. Range-dependent eigenvalue and eigenfunctions

In this subsection, we consider the range-dependent problem. For fixed ω, we assume Eq.(11) can be rewritten as

$$\rho_0(z)\left[\frac{\partial}{\partial z}\frac{1}{\rho_0(z)}\frac{\partial}{\partial z}\right]\varphi_n(x,z)$$
$$+\left[\frac{\omega^2}{c_0(z)^2}+\beta(z,\omega)x\right]\varphi_n(x,z)=k_n^2\varphi_n(x,z). \quad (18)$$

This equation is satisfied when the range-dependent sound speed perturbation can be considered linearly proportional to the range parameter x.

Defining

$$H_0 \equiv \rho_0\left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\rho_0}\frac{\mathrm{d}}{\mathrm{d}z}\right)\right]+\frac{\omega_0^2}{c_0(z)^2}, \quad (19)$$

$$H(x) \equiv H_0+\omega^2\beta x, \quad (20)$$

treating $t=x$ as the "time" variable, and applying the same algorithm above, we can find $\varphi_n(n=1,2,\cdots,N)$ as functions of variable x in the same way as the broad band problem.

3.3. Evolution of normal mode with environmental parameters

Sound propagation in the ocean is very complicated. The environment changes with time and space. Although many endeavours have been made to solve the sound propagation problem in a complicated ocean environment, the present understanding is still poor. Here we show that the normal mode method can be useful for solving the problem.

As is usually done in the articles, we assume that the range-dependent sound speed distribution can be approximated by a set of so-called empirical orthogonal functions

$$C(x,z)=\overline{C}(z)+\sum_{i=1}^{m}a_i f_i(x,z), \quad (21)$$

where the coefficients $a_i(i=1,2,\cdots,M)$ are the expanding coefficients. The question we want to answer is how the eigenvalues and eigenfunctions change with the expanding coefficients.

Inserting Eq.(21) into Eq.(11) we obtain

$$\rho_0(z)\left[\frac{\partial}{\partial z}\frac{1}{\rho_0(z)}\frac{\partial}{\partial z}\right]\varphi(x,z)+\omega^2\left[\frac{1}{\overline{C}(z)^2}\right.$$
$$\left.-2\frac{1}{\overline{C}^3}\sum_{i=1}^{M}a_i f_i(x,z)\right]\varphi(x,z)=k^2\varphi(x,z). \quad (22)$$

Defining

$$H_0 = \rho_0(z)\left[\frac{\partial}{\partial z}\frac{1}{\rho_0(z)}\frac{\partial}{\partial z}\right] + \omega^2\left[\frac{1}{\overline{C(z)^2}}\right] \tag{23}$$

and

$$H_1 = H_0 + \sum_{i=1}^{M} a_i V_i(x,z), \quad V_i = -2\frac{\omega^2}{\overline{C}^3} f_i(x,z). \tag{24}$$

Equation (24) has a similar form as Eq.(14), but has M strength parameters instead of a single strength parameter.

In this case, system (2) is replaced by the multi-time dynamic system

$$\frac{\mathrm{d}}{\mathrm{d}t_i}x_n(\boldsymbol{t}) = p_n(\boldsymbol{t}) \equiv \langle n(\boldsymbol{t})|V_i|n(\boldsymbol{t})\rangle, \tag{25a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t_i}p_n(\boldsymbol{t}) = 2\sum_{m(\neq n)}[V_i(\boldsymbol{t})]_{mn}[V_i(\boldsymbol{t})]_{mn}(x_n - x_m)^{-1}, \tag{25b}$$

$$\frac{\mathrm{d}|n(\boldsymbol{t})\rangle}{\mathrm{d}t_i} = \sum_{m(\neq n)}|m\rangle[V_i]_{mn}(x_n - x_m)^{-1}, \tag{25c}$$

$$\frac{\mathrm{d}\langle n(\boldsymbol{t})|}{\mathrm{d}t_i} = \sum_{m(\neq n)}\langle m(\boldsymbol{t})|[V_i]_{mn}(x_n - x_m)^{-1}, \tag{25d}$$

where $\boldsymbol{t} = (t_1, t_2, \cdots, t_M)$, $t_i = a_i (i = 1, 2, \cdots, M)$. Using this system we can investigate the evolution of normal modes with the expanding coefficients.

4. Numerical example for broad band problem

To test the method given here, we consider a numerical example of the broad band problem for the shallow water case. The environmental model is a Pekeris's shallow water propagation problem as shown in Fig.1. The sound speed and density in the water column are 1500.0 m/s and 1.0 g/cm^3, 1623.0m/s and 1.57 g/cm^3 in the half space sea bed, respectively.

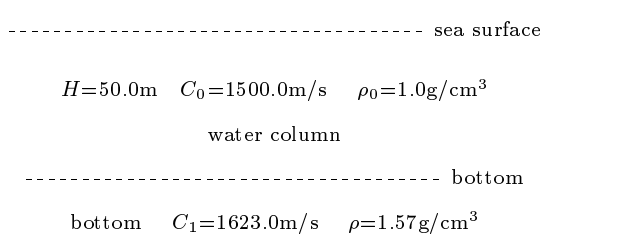


Fig.1. Acoustic model used in simulation.

The numerical simulation is to interpolate a horizontal constant from a single frequency to broad band frequencies. The reference frequency f_0 is taken to be 1.0kHz, and the eigenvalues and eigenfunctions for the reference frequency are calculated by the KRAKAN code. The total number of modes with phase speed smaller than the sound speed of the sea bed is 88, so the dimension of the matrix $[H]$ defined in Eq.(16) is N=88. The eigenvalues for the other frequencies are obtained by diagonalizing $[X(0) + P(0)t]$ defined in Eq.(11). The difference of the horizontal wavenumber between the numerical results based on the present method and the results obtained by the KRAKAN code is shown in Figs.2(a) and 2(b). It is seen that for frequencies 5.0 and 10.0 kHz, the normal mode wavenumber for the first 30 modes calculated by the present method and the KRAKAN code agree well down to the 10^{-5} order, and the normal mode wavenumbers for the first ten modes agree well even down to 10^{-6}.

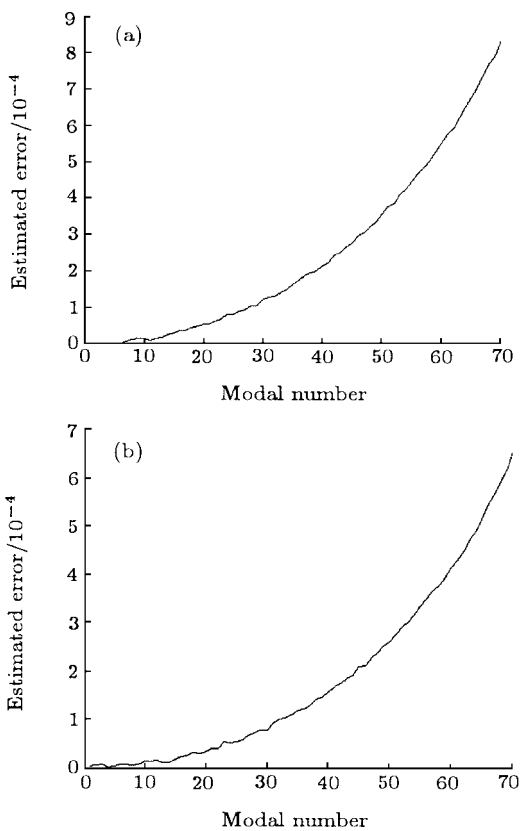


Fig.2. Estimated eigenvalue differences between the presented method and the KRAKAN code: frequency=5.0kHz(a), 10.0kHz(b), reference frequency=1.0kHz.

5. Discussion and summary

In this paper, we have proposed an alternate approach to the normal mode problem. The method presented here is in fact equivalent to directly solving eigenvalue problems, but provides another way to

solve a special type of eigenvalue problem by using the dynamic system. A simple numerical test for the broad band problem shows that the method is effective. The method presented has advantages over the usual code in three respects.

1. For the broad band problem, in general, the computational cost of a modal eigenvalue solver is determined by the product of vertical layer number by frequency. Meanwhile, one has to take greater layer numbers for high frequencies at each calculation step. However, the method presented here does not need to do this if we interpolate the mode from the high-frequency side to the low-frequency side. The only cost that must be paid is firstly to calculate the normal mode problem for one high enough frequency, then project the problem into a matrix problem. The method presented here may also be useful for the broad band problem of sonar signal processing.

2. The programming is straightforward, and the same code can be used for both broad band and range-dependent problems provided the range-dependent perturbation can be approximated to the form of Eq.(18).

3. By transforming a normal mode problem into a dynamic system problem and treating the environmental parameters as "time " variables, the eigenvalue evolution can be imaged as "particle" trace with the "time " change. This will be very helpful for understanding the acoustic interaction under a complicated ocean environment.

This paper only present our main ideas. Further numerical tests and practical applications will be considered in the near future.

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