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Noether's theorem of a rotational relativistic variable mass system

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Noether's theory of a rotational relativistic variable mass system is studied. Firstly, Jourdain's principle of the rotational relativistic variable mass system is given. Secondly, on the basis of the invariance of the Jourdain's principle under the infinitesimal transformations of groups, Noether's theorem and its inverse theorem of the rotational relativistic variable mass system are presented. Finally, an example is given to illustrate the application of the result.

Keywords: relativity, analytical mechanics, rotational variable mass system, Noether's theorem
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1. Introduction

The study of symmetry and conservation law of mechanical systems is an important topic of modern analytical mechanics. In 1918, using the invariance of the Hamilton action under the infinitesimal transformations, Noether[1] studied the symmetries of a mechanical system and formulated the famous Noether theorem. The symmetries and conservation laws can also be studied by using differential variational principles.[2] In recent years, a series of important results have been obtained on the study of the Noether symmetry and conservation law of classical mechanical systems.[3−11] With the progress of science and technology, more and more experimental phenomena are related to high-speed rotation. The relativistic effect must be considered when studying the high-speed rotation question. In 1985, Carmeli[12,13] set up the theory of rotational relativistic mechanics. The dynamical theory of the rotational relativistic constant mass system has been studied in Refs.[14]–[18]. The dynamical equation and Lie symmetry theory of the rotational relativistic variable mass system have been studied in Refs.[19] and [20]. In this paper we study the Noether symmetry and conservation laws of a rotational relativistic variable mass system by means of the differential variational principle. The Noether theorem and its inverse theorem of the rotational relativistic variable mass system are presented. An example is given to illustrate the application of the result.

2. Jourdain's principle of the rotational relativistic variable mass system

Consider a mechanical system consisting of N variable mass particles. Suppose at time t, one particle, whose rest mass is m_{oi}, rotates about axis oz with angular velocity ω_i, its distance from the axis r_i is rigidly invariable, and its classical moment of inertia $I_{oi} = r_i^2 m_{oi}$; and another small particle, whose rest mass is Δm_{oi}, rotates about axis oz with angular velocity Ω_i, and its classical moment of inertia $\Delta I_{oi} = r_i^2 \Delta m_{oi}$. They combine with each other during the time interval Δt after t. Then the relativistic fundamental dynamical equation of the system is[19]

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_i\omega_i) - \frac{\dot{I}_{oi}\omega_i}{\sqrt{1-\omega_i^2/\Gamma_i^2}} = M_i + M_i' + M_i^* \qquad (i = 1, 2, \cdots, N), \tag{1}$$

where $I_i = \dfrac{I_{oi}}{\sqrt{1-\omega_i^2/\Gamma_i^2}}$ is the relativistic moment of inertia, $\Gamma_i = \sqrt{\dfrac{E_{oi}}{I_{oi}}} = \dfrac{c}{r_i}$ (where $E_{oi} = m_{oi}c^2$) is the limiting angular velocity,[12] $M_i^* = \dot{I}_{oi}\left(\dfrac{\Omega_i}{\sqrt{1-\Omega_i^2/\Gamma_i^2}} - \dfrac{\omega_i}{\sqrt{1-\omega_i^2/\Gamma_i^2}}\right)$ is the moment of relativistic reacting thrust, M_i is the moment of active force and M_i' is the moment of constraint force.

Introducing the virtual displacement $\delta_1\dot{\theta}_i$ (where θ_i are angular coordinates, $\dot{\theta}_i = \omega_i$) of the angu-

lar velocity space, similar to the derivation of the d'Alembert principle of the rotational relativistic variable mass system in Ref.[20], we get Jourdain's principle of the rotational relativistic variable mass system as follows

$$\sum_{s=1}^{n}\left[-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T_r^*}{\partial\dot{q}_s}+\frac{\partial T_r^*}{\partial q_s}+G_s+G_s^*\right]\delta_1\dot{q}_s=0, \tag{2}$$

where

$$T_r^*=\sum_{i=1}^{N}I_{oi}\Gamma_i^2\left(1-\sqrt{1-\omega_i^2/\Gamma_i^2}\right), \tag{3}$$

$$\begin{aligned}G_s^*=&\sum_{i=1}^{N}\frac{\dot{I}_{oi}\Omega_i}{\sqrt{1-\Omega_i^2/\Gamma_i^2}}\frac{\partial\theta_i}{\partial q_s}\\&-\sum_{i=1}^{N}\frac{\partial I_{oi}}{\partial q_s}\Gamma_i^2\left(1-\sqrt{1-\omega_i^2/\Gamma_i^2}\right)\\&+\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial I_{oi}}{\partial\dot{q}_s}\Gamma_i^2\left(1-\sqrt{1-\omega_i^2/\Gamma_i^2}\right)\right],\end{aligned} \tag{4}$$

$$G_s=\sum_{i=1}^{N}M_i\frac{\partial\theta_i}{\partial q_s}. \tag{5}$$

If the generalized forces G_s can be divided into G_s' with potential energy and G_s'' without potential energy, we have

$$G_s'=-\frac{\partial V}{\partial q_s},\quad G_s=-\frac{\partial V}{\partial q_s}+G_s''. \tag{6}$$

Then Eq.(2) can be written in the form

$$\sum_{s=1}^{n}\left[-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r^*}{\partial\dot{q}_s}+\frac{\partial L_r^*}{\partial q_s}+G_s''+G_s^*\right]\delta_1\dot{q}_s=0, \tag{7}$$

where $L_r^*=T_r^*-V$ is the generalized Lagrangian of the rotational relativistic variable mass system.

3. Jourdain's variation

Defining the Jourdain non-simultaneous variation $\Delta_1\dot{q}_s,(\Delta_1 q_s)^{\cdot}$ and $(\Delta_1 t)^{\cdot}$, it is required

$$\Delta_1 q_s=0,\quad \Delta_1 t=0. \tag{8}$$

Using the relation between Δ_1 and δ_1, it is easy to verify that[2]

$$\Delta_1\dot{q}_s=\delta_1\dot{q}_s+\ddot{q}_s\Delta_1 t, \tag{9}$$

$$(\Delta_1 q_s)^{\cdot}=\delta_1\dot{q}_s+\ddot{q}_s\Delta_1 t+\dot{q}_s(\Delta_1 t)^{\cdot}, \tag{10}$$

$$\Delta_1\dot{q}_s=(\Delta_1 q_s)^{\cdot}-\dot{q}_s(\Delta_1 t)^{\cdot}. \tag{11}$$

We make the infinitesimal transformations of generalized coordinates, time and generalized velocities[2] as follows

$$\begin{cases}q_s^*=q_s,\quad t^*=t,\\ \dfrac{\mathrm{d}q_s^*}{\mathrm{d}t}-\dfrac{\mathrm{d}q_s}{\mathrm{d}t}=\delta_1\dot{q}_s=\Delta_1\dot{q}_s,\\ \dfrac{\mathrm{d}q_s^*}{\mathrm{d}t^*}-\dfrac{\mathrm{d}q_s}{\mathrm{d}t}=(\Delta_1 q_s)^{\cdot}.\end{cases} \tag{12}$$

Considering the infinitesimals $(\Delta_1 q_s)^{\cdot}$ and $(\Delta_1 t)^{\cdot}$ as the composition unit for Jourdain infinitesimal transformations and introducing the Jourdain generators of space and time, we have

$$(\Delta_1 q_s)^{\cdot}=\varepsilon F_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad (\Delta_1 t)^{\cdot}=\varepsilon f(t,\boldsymbol{q},\dot{\boldsymbol{q}}). \tag{13}$$

Using Eqs.(13), the Jourdain simultaneous variation can be expressed, by those generators, as

$$\delta_1\dot{q}_s=\varepsilon[F_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})-\dot{q}_s f(t,\boldsymbol{q},\dot{\boldsymbol{q}})]. \tag{14}$$

4. Noether's theorem of the rotational relativistic variable mass system

Substituting formula (14) into Eq.(7), we obtain

$$\sum_{s=1}^{n}\varepsilon\left[-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r^*}{\partial\dot{q}_s}+\frac{\partial L_r^*}{\partial q_s}+G_s''+G_s^*\right](F_s-\dot{q}_s f)=0. \tag{15}$$

Adding and subtracting a function $\varepsilon\dfrac{\partial L_r^*}{\partial t}f$ from Eq.(15), we obtain

$$\begin{aligned}&\varepsilon\Bigg\{\sum_{s=1}^{n}\left[(G_s''+G_s^*)(F_s-\dot{q}_s f)+\frac{\partial L_r^*}{\partial q_s}F_s+\frac{\partial L_r^*}{\partial\dot{q}_s}\dot{F}_s\right]\\&-\left(\frac{\partial L_r^*}{\partial t}+\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial q_s}\dot{q}_s+\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial\dot{q}_s}\ddot{q}_s\right)f+\frac{\partial L_r^*}{\partial t}f\\&-\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial\dot{q}_s}\dot{q}_s\dot{f}-\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial\dot{q}_s}(F_s-\dot{q}_s f)\right]\Bigg\}=0.\end{aligned} \tag{16}$$

Adding and subtracting a gauge function $\varepsilon\dot{P}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ from Eq.(16), we obtain

$$\begin{aligned}&\varepsilon\Bigg\{\sum_{s=1}^{n}\left[(G_s''+G_s^*)(F_s-\dot{q}_s f)+\frac{\partial L_r^*}{\partial q_s}F_s+\frac{\partial L_r^*}{\partial\dot{q}_s}\dot{F}_s\right]\\&+\left(L_r^*-\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial\dot{q}_s}\dot{q}_s\right)\dot{f}+\frac{\partial L_r^*}{\partial t}f+\dot{P}(t,\boldsymbol{q},\dot{\boldsymbol{q}})\\&-\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial\dot{q}_s}F_s+\left(L_r^*-\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial\dot{q}_s}\dot{q}_s\right)f\right.\\&\left.+P(t,\boldsymbol{q},\dot{\boldsymbol{q}})\right]\Bigg\}=0.\end{aligned} \tag{17}$$

Equation (17) is the condition of invariance for the Jourdain's principle of the rotational relativistic variable mass system.

If generators f, F_s of infinitesimal transformations and gauge function $P(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy

$$\sum_{s=1}^{n}\Big[(G_s''+G_s^*)(F_s-\dot{q}_s f) + \frac{\partial L_r^*}{\partial q_s}F_s+\frac{\partial L_r^*}{\partial \dot{q}_s}\dot{F}_s\Big]+\Big(L_r^*-\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial \dot{q}_s}\dot{q}_s\Big)\dot{f} + \frac{\partial L_r^*}{\partial t}f+\dot{P}(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0, \tag{18}$$

then we can get conserved quantity from Eq.(17) as follows

$$\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial \dot{q}_s}F_s+\Big(L_r^*-\sum_{s=1}^{n}\frac{\partial L_r^*}{\partial \dot{q}_s}\dot{q}_s\Big)f+P(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\text{const.} \tag{19}$$

Equation (18) is called Noether's identity of the rotational relativistic variable mass system.

Theorem 1. For the rotational relativistic variable mass holonomic system, if generators f, F_s and gauge function $P(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy condition (18), then the system has conservation law in the form of Eq.(19).

Suppose the system is subjected to g ideal nonholonomic constraints

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0 \qquad (\beta=1,2,\cdots,g), \tag{20}$$

the restriction which constraints (20) impose on the virtual displacement $\delta_1\dot{q}_s$ is

$$\sum_{s=1}^{n}\frac{\partial f_\beta}{\partial \dot{q}_s}\delta_1\dot{q}_s=0 \qquad (\beta=1,2,\cdots,g). \tag{21}$$

Substituting Eqs.(14) into Eq.(21), and noting the randomicity of parameter ε, we obtain

$$\sum_{s=1}^{n}\frac{\partial f_\beta}{\partial \dot{q}_s}(F_s-\dot{q}_s f)=0. \tag{22}$$

This is the limit of nonholonomic constraints (20) on infinitesimal generators. Then we have

Theorem 2. For the rotational relativistic variable mass nonholonomic system, if generators f, F_s and gauge function $P(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfy conditions (18) and (22), then the system has conservation law in the form of Eq.(19).

Theorems 1 and 2 are called the generalized Noether's theorem of the rotational relativistic variable mass holonomic system and the rotational relativistic variable mass nonholonomic system, respectively. They present the conditions and form of the existence of conservation law for the rotational relativistic variable mass holonomic and nonholonomic systems.

5. Noether's inverse theorem of the rotational relativistic variable mass system

Suppose that a first integral of the rotational relativistic variable mass system has been given as

$$I=I(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\text{const.}, \tag{23}$$

then we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+\sum_{s=1}^{n}\frac{\partial I}{\partial q_s}\dot{q}_s+\sum_{s=1}^{n}\frac{\partial I}{\partial \dot{q}_s}\ddot{q}_s=0. \tag{24}$$

Equation (7) can be written in the form

$$\sum_{s=1}^{n}\Big(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_r^*}{\partial \dot{q}_s}+\frac{\partial L_r^*}{\partial q_s}+G_s''+G_s^*\Big)\overline{F}_s=0, \tag{25}$$

where

$$\overline{F}_s=F_s-\dot{q}_s f. \tag{26}$$

According to Eq.(4), G_s^* is generally a linear function of $\ddot{q}$ and can be written as

$$G_s^*=\sum_{k=1}^{n}W_{sk}^*(t,\boldsymbol{q},\dot{\boldsymbol{q}})\ddot{q}_k+W_s^*(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{27}$$

where

$$W_{sk}^*=\sum_{i=1}^{N}\Big[\frac{\partial I_{oi}}{\partial \dot{q}_k}\frac{\Omega_i}{\sqrt{1-\Omega_i^2/\Gamma_i^2}}\frac{\partial \theta_i}{\partial q_s} + \frac{\partial^2 I_{oi}}{\partial \dot{q}_s\partial \dot{q}_k}\Gamma_i^2\Big(1-\sqrt{1-\omega_i^2/\Gamma_i^2}\Big) + \frac{\partial I_{oi}}{\partial \dot{q}_s}\frac{\omega_i}{\sqrt{1-\omega_i^2/\Gamma_i^2}}\frac{\partial \theta_i}{\partial q_k}\Big]. \tag{28}$$

Adding Eq.(24) to Eq.(25), and putting the coefficients of $\ddot{q}_k$ equal to zero, we obtain

$$\sum_{s=1}^{n}\Big(\frac{\partial^2 L_r^*}{\partial \dot{q}_s\partial \dot{q}_k}-W_{sk}^*\Big)\overline{F}_s-\frac{\partial I}{\partial \dot{q}_k}=0. \tag{29}$$

Suppose

$$\det(h_{sk})=\det\Big(\frac{\partial^2 L_r^*}{\partial \dot{q}_s\partial \dot{q}_k}-W_{sk}^*\Big)\neq 0, \tag{30}$$

then from Eq.(29) we obtain

$$\overline{F}_s = \sum_{k=1}^{n} \tilde{h}_{sk} \frac{\partial I}{\partial \dot{q}_k}, \tag{31}$$

in which

$$\tilde{h}_{sk} h_{kl} = \delta_{sl}, \quad h_{sk} = \frac{\partial^2 L_r^*}{\partial \dot{q}_s \partial \dot{q}_k} - W_{sk}^*. \tag{32}$$

Now let the integral (23) equal conserved quantity (19), i.e.

$$I = \sum_{s=1}^{n} \frac{\partial L_r^*}{\partial \dot{q}_s} F_s + \left(L_r^* - \sum_{s=1}^{n} \frac{\partial L_r^*}{\partial \dot{q}_s} \dot{q}_s \right) f + P(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{33}$$

Thus, from Eqs.(31) and (33), generators f, F_s of infinitesimal transformation can be found.

Theorem 3. If the integral (23) of the rotational relativistic variable mass holonomic system has been given, then the infinitesimal transformations determined by Eqs.(12), (13), (31) and (33) are the system's transformation satisfying Noether's identity (18).

Theorem 4. If the integral (23) of rotational relativistic variable mass nonholonomic system has been given, then the infinitesimal transformations determined by Eqs.(12), (13), (31) and (33) are the system's transformation satisfying Noether's identity (18) as long as they satisfy condition (22).

Theorem 3 is called the generalized Noether's inverse theorem of the rotational relativistic variable mass holonomic system, and theorem 4 is called the generalized Noether's inverse theorem of the rotational relativistic variable mass nonholonomic system.

6. Illustrative example

Suppose the Lagrangian of a rotational relativistic variable mass system is

$$L_r^* = k\mathrm{e}^{-\alpha t} \Gamma^2 \left(1 - \sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2} \right) - V$$

$$(k, \alpha \text{ are constants}), \tag{34}$$

the non-potential generalized forces are $G_\theta'' = \dot{\theta}$ and $G_\varphi'' = -\dot{\varphi}$, potential energy V is constant, $\Omega = 0$. Let us study the Noether's symmetries and conserved quantities of the system.

Firstly, let us seek the conserved quantities from the given symmetry. By virtue of Eq.(18), the generators of the infinitesimal transformation satisfy

$$\begin{aligned} &\dot{\theta}(F_1 - \dot{\theta} \dot{f}) - \dot{\varphi}(F_2 - \dot{\varphi} f) \\ &+ \frac{k\mathrm{e}^{-\alpha t}}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} (\dot{\theta} \dot{F}_1 + \dot{\varphi} \dot{F}_2) \\ &- \alpha k \mathrm{e}^{-\alpha t} \Gamma^2 \left(1 - \sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2} \right) f \\ &+ k\mathrm{e}^{-\alpha t} \left[\Gamma^2 \left(1 - \sqrt{1 - (\dot{\theta} + \dot{\varphi})/\Gamma^2} \right) \right. \\ &\left. - \frac{\dot{\theta}^2 + \dot{\varphi}^2}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} \right] \dot{f} - V \dot{f} + \dot{P} = 0. \end{aligned} \tag{35}$$

We can find solutions of Eq.(35) as follows

$$f = 0, \quad F_1 = 0, \quad F_2 = 1, \quad P = \varphi, \tag{36}$$

$$f = 0, \quad F_1 = 1, \quad F_2 = 0, \quad P = -\theta, \tag{37}$$

$$f = 0, \quad F_1 = -\varphi, \quad F_2 = \theta, \quad P = \theta\varphi. \tag{38}$$

According to theorem 1, we can get conserved quantities of the system for Eqs.(36), (37) and (38), respectively, as follows

$$I_1 = k\mathrm{e}^{-\alpha t} \frac{\dot{\varphi}}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} + \varphi = \text{const.}, \tag{39}$$

$$I_2 = k\mathrm{e}^{-\alpha t} \frac{\dot{\theta}}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} - \theta = \text{const.}, \tag{40}$$

$$I_3 = k\mathrm{e}^{-\alpha t} \frac{\theta\dot{\varphi} - \varphi\dot{\theta}}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} + \theta\varphi = \text{const.} \tag{41}$$

Secondly, let us find the corresponding infinitesimal transformations from a known integral. Suppose there is an integral in the form

$$I = I_2 = k\mathrm{e}^{-\alpha t} \frac{\dot{\theta}}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} - \theta, \tag{42}$$

then we have

$$\begin{aligned} \frac{\partial I}{\partial \dot{\theta}} =& k\mathrm{e}^{-\alpha t} \left[\frac{1}{\sqrt{1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2}} \right. \\ &\left. + \frac{\dot{\theta}^2}{\Gamma^2 [1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2]^{3/2}} \right], \end{aligned} \tag{43}$$

$$\frac{\partial I}{\partial \dot{\varphi}} = k\mathrm{e}^{-\alpha t} \frac{\dot{\theta}\dot{\varphi}}{\Gamma^2 [1 - (\dot{\theta}^2 + \dot{\varphi}^2)/\Gamma^2]^{3/2}}, \tag{44}$$

$$h_{11} = \frac{\partial^2 L_r^*}{\partial \dot{\theta}^2} = \frac{\partial I}{\partial \dot{\theta}}, \tag{45}$$

$$h_{12} = \frac{\partial^2 L_r^*}{\partial \dot{\theta} \partial \dot{\varphi}} = \frac{\partial I}{\partial \dot{\varphi}}, \tag{46}$$

$$h_{21} = \frac{\partial^2 L_r^*}{\partial \dot{\varphi} \partial \dot{\theta}} = \frac{\partial I}{\partial \dot{\varphi}} = h_{12}, \tag{47}$$

$$h_{22} = \frac{\partial^2 L_r^*}{\partial \dot{\varphi}^2} = k\mathrm{e}^{-\alpha t}\left[\frac{1}{\sqrt{1-(\dot{\theta}^2+\dot{\varphi}^2)/\Gamma^2}} + \frac{\dot{\varphi}^2}{\Gamma^2[1-(\dot{\theta}^2+\dot{\varphi}^2)/\Gamma^2]^{3/2}}\right]. \quad (48)$$

From Eq.(31) we obtain

$$\overline{F}_1 = F_1 - \dot{\theta} f = \tilde{h}_{11}\frac{\partial I}{\partial \dot{\theta}} + \tilde{h}_{12}\frac{\partial I}{\partial \dot{\varphi}} = \tilde{h}_{11}h_{11} + \tilde{h}_{21}h_{12} = 1, \quad (49)$$

$$\overline{F}_2 = F_2 - \dot{\varphi} f = \tilde{h}_{21}\frac{\partial I}{\partial \dot{\theta}} + \tilde{h}_{22}\frac{\partial I}{\partial \dot{\varphi}} = \tilde{h}_{21}h_{11} + \tilde{h}_{22}h_{21} = 0. \quad (50)$$

Equation (33) gives

$$k\mathrm{e}^{-\alpha t}\left[\frac{\dot{\theta}F_1 + \dot{\varphi}F_2}{\sqrt{1-(\dot{\theta}^2+\dot{\varphi}^2)/\Gamma^2}} + \Gamma^2\left(1-\sqrt{1-(\dot{\theta}^2+\dot{\varphi}^2)/\Gamma^2}\right) f - \frac{\dot{\theta}^2+\dot{\varphi}^2}{\sqrt{1-(\dot{\theta}^2+\dot{\varphi}^2)/\Gamma^2}} f\right] - Vf + P = k\mathrm{e}^{-\alpha t}\frac{\dot{\theta}}{\sqrt{1-(\dot{\theta}^2+\dot{\varphi}^2)/\Gamma^2}} - \theta. \quad (51)$$

Taking

$$P = -\theta, \quad (52)$$

from Eqs.(49), (50) and (51), we can get the following solution

$$f = 0, \quad F_1 = 1, \quad F_2 = 0. \quad (53)$$

References

[1] Noether A E 1918 *Math. Phys.* KI **II** 235
[2] Vujanovic B 1986 *Acta Mech.* **65** 63
[3] Zhang J F 1989 *Chin. Sci. Bull.* **34** 1756 (in Chinese)
[4] Liu D 1990 *Sci. China* A **20** 1189 (in Chinese)
[5] Li Z P 1992 *Chin. Sci. Bull.* **37** 2204 (in Chinese)
[6] Mei F X 1993 *Sci. China* A **36** 1456
[7] Zhao Y Y and Mei F X 1993 *Adv. Mech.* **23** 360 (in Chinese)
[8] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[9] Fang J H 2000 *Appl. Math. Mech.* **21** 836
[10] Zhang Y and Mei F X 2000 *Appl. Math. Mech.* **21** 53
[11] Li Y C, Zhang Y, Liang J H and Mei F X 2001 *Chin. Phys.* **10** 376
[12] Carmeli M 1985 *Found. Phys.* **15** 889
[13] Carmeli M 1986 *Int. J. Theor. Phys.* **25** 89
[14] Luo S K 1998 *Appl. Math. Mech.* **19** 43 (in Chinese)
[15] Luo S K, Fu J L and Chen X W 2001 *Acta Phys. Sin.* **50** 383 (in Chinese)
[16] Qiao Y F, Li R J and Meng J 2001 *Acta Phys. Sin.* **50** 1637 (in Chinese)
[17] Luo S K, Guo Y X, Chen X W and Fu J L 2001 *Acta Phys. Sin.* **50** 2049 (in Chinese)
[18] Luo S K, Guo Y X and Chen X W 2001 *Acta Phys. Sin.* **50** 2053 (in Chinese)
[19] Fang J H 2000 *Acta Phys. Sin.* **49** 1028 (in Chinese)
[20] Fang J H 2001 *Acta Phys. Sin.* **50** 390 (in Chinese)