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Chaos in the second-order autonomous Birkhoff system with a heteroclinic circle*

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Chaotic behaviour in a second-order autonomous Birkhoff system with a heteroclinic circle under weakly periodic perturbation is studied using the Melnikov method. The equations of heteroclinic orbits and the criteria for chaos are given. One example is also presented to illustrate the application of the results.

Keywords: Melnikov's method, heteroclinic circle, Birkhoff system, chaos
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1. Introduction

The dynamics of the Birkhoff system is an important branch of classical mechanics in modern times. The Birkhoff system was proposed in the well-known work "Dynamical Systems" written by the American mathematician Birkhoff.[1] The American physicist Stantilli carried out primary research on the Birkhoff system and gained some results.[2] After that, Ref.[3] firstly constructed the frame of Birkhoff dynamics. In recent years, the study of Birkhoff dynamics has become a popular subject and many important results have been obtained.[4−13] But there is still no global analysis for the Birkhoff system, which includes global stability, bifurcation and chaos.

Melnikov's method is an important method of analysis in investigating chaos. Many physical and mechanical problems can be classified to those problems of second-order ordinary differential equations with weakly periodic perturbation and homoclinic orbits or heteroclinic circles.[14,15] For these systems, we can construct two-dimensional (2D) Poincaré mapping by certain skills. Melnikov's method is exactly an analytic method being used to judge whether the 2D Poincaré mapping has a Smale horseshoe transformation or not. By the theory of dynamical systems, if a plane mapping has a Smale horseshoe transformation, the mapping has the invariant set showing chaotic behaviour. So, Melnikov's method can be used to judge chaos in the sense of Smale horseshoes. In this paper, the chaotic behaviour in a second-order autonomous Birkhoff system with weakly periodic perturbation and heteroclinic circles is studied using Melnikov's method.

2. Equations of motion and equations of heteroclinic orbit

The equations of a second-order Birkhoff system are[2,3]

$$\dot{a}^1 = \omega^{12}\left(\frac{\partial B}{\partial a^2} + \frac{\partial R_2}{\partial t}\right), \quad \dot{a}^2 = \omega^{21}\left(\frac{\partial B}{\partial a^1} + \frac{\partial R_1}{\partial t}\right), \tag{1}$$

where

$$\omega^{12} = -\omega^{21} = (\omega_{12})^{-1} = \left(\frac{\partial R_2}{\partial a^1} - \frac{\partial R_1}{\partial a^2}\right)^{-1}. \tag{2}$$

If the Birkhoff functions R_1 and R_2 and the Birkhoffian B do not include the time t explicitly, the system is called an autonomous system. For this case, Eqs.(1) reduce to a simple form

$$\dot{a}^1 = \omega^{12}\frac{\partial B}{\partial a^2}, \quad \dot{a}^2 = \omega^{21}\frac{\partial B}{\partial a^1}. \tag{3}$$

From Ref.[16] we know that the second-order autonomous Birkhoff system may have saddle points, cusps and centres. Suppose that the system has n hyperbolic saddle points $\{p_1, p_2, \cdots, p_n\}$ and some heteroclinic orbits $\Gamma_i(a_i^1(t), a_i^2(t))$ linking p_i to p_{i+1} (where i=1,2,$\cdots$, n, n+1 = 1), where the n heteroclinic orbits form a heteroclinic circle. How to find the equations

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of the n heteroclinic orbits? In fact, the derivative of the Birkhoffian B with respect to time t by Eqs.(3) is

$$\begin{aligned}\frac{\mathrm{d}B}{\mathrm{d}t} =& \frac{\partial B}{\partial a^1}\dot{a}^1 + \frac{\partial B}{\partial a^2}\dot{a}^2 \\ =& \frac{\partial B}{\partial a^1}\omega^{12}\frac{\partial B}{\partial a^2} + \frac{\partial B}{\partial a^2}\omega^{21}\frac{\partial B}{\partial a^1} \equiv 0. \end{aligned} \tag{4}$$

Hence, $B(a^1, a^2) = h$ is an integral curve of the system. If the system has heteroclinic orbits, then equations of the heteroclinic orbits are determined by $B(a^1, a^2) = h$. Considering this and Eq.(3) at the same time, we can obtain the parametric equations $\Gamma_i(a_i^1(t), a_i^2(t))$ of the heteroclinic orbits. So we have:

Proposition 1. If the second-order autonomous Birkhoff system has heteroclinic orbits, the equations of heteroclinic orbits are determined by $B(a^1, a^2) = h$, and their parametric equations $\Gamma_i(a_i^1(t), a_i^2(t))$ with respect to time t can be obtained by combining $B(a^1, a^2) = h$ and Eq.(3).

3. Criteria for chaos

We consider the perturbed system as follows

$$\begin{aligned}\dot{a}^1 &= \omega^{12}\frac{\partial B}{\partial a^2} + \varepsilon g_1(a^1, a^2, \lambda, t), \\ \dot{a}^2 &= \omega^{21}\frac{\partial B}{\partial a^1} + \varepsilon g_2(a^1, a^2, \lambda, t), \end{aligned} \tag{5}$$

where $\omega^{12}\dfrac{\partial B}{\partial a^2}$, $\omega^{21}\dfrac{\partial B}{\partial a^1}$, $g_1(a^1, a^2, \lambda, t)$, $g_2(a^1, a^2, \lambda, t)$ are sufficiently smooth C^r-functions ($r \geq 2$), and they are bounded in some bounded sets, $g_1(a^1, a^2, \lambda, t)$ and $g_2(a^1, a^2, \lambda, t)$ are T-periodic functions of time t.

If we construct the suspended system of Eq.(5):

$$\begin{aligned}\dot{a}^1 &= \omega^{12}\frac{\partial B}{\partial a^2} + \varepsilon g_1(a^1, a^2, \lambda, \theta), \\ \dot{a}^2 &= \omega^{21}\frac{\partial B}{\partial a^1} + \varepsilon g_2(a^1, a^2, \lambda, \theta), \quad \theta \in S', \\ \dot{\theta} &= 1, \end{aligned} \tag{6}$$

and if S' is a circle with the perimeter T, then in the global cross section we have

$$\Sigma^{t_0} = \{(a^1, a^2, \theta) \in R \times R \times S' | \theta = t_0 \in [0, T]\}. \tag{7}$$

We discuss the Poincaré mapping $P_\varepsilon^{t_0} : \Sigma^{t_0} \to \Sigma^{t_0}$ of system (6). For ε small enough, $P_\varepsilon^{t_0}$ also has n hyperbolic saddle points $p_{i,\varepsilon}^{t_0} = p_i + o(\varepsilon)$ (where i=1,2,$\cdots$, n). If the stable and unstable manifolds of the n hyperbolic saddle points form an n-transverse ring, then it is sure that chaos exists in the sense of Smale horseshoe.

By using the Melnikov functions of non-Hamiltonian systems given in Ref.[14], and considering

$$\mathrm{Tr}\boldsymbol{D}\begin{pmatrix} \omega^{12}\dfrac{\partial B}{\partial a^2} \\ \omega^{21}\dfrac{\partial B}{\partial a^1} \end{pmatrix} = \frac{\partial \omega^{12}}{\partial a^1}\frac{\partial B}{\partial a^2} - \frac{\partial \omega^{12}}{\partial a^2}\frac{\partial B}{\partial a^1},$$

we can construct n Melnikov functions corresponding to the n heteroclinic orbits

$$\begin{aligned} M_i(t_0) = \int_{-\infty}^{+\infty} & \left[\left(\omega^{12}\frac{\partial B}{\partial a^2}\right)_i g_2(a_i^1(t), a_i^2(t), \lambda, t + t_0) \right. \\ & \left. - \left(\omega^{21}\frac{\partial B}{\partial a^1}\right)_i g_1(a_i^1(t), a_i^2(t), \lambda, t + t_0)\right] \\ & \times \exp\left[-\int_0^t \left(\frac{\partial \omega^{12}}{\partial a^1}\frac{\partial B}{\partial a^2} - \frac{\partial \omega^{12}}{\partial a^2}\frac{\partial B}{\partial a^1}\right)_i \mathrm{d}t\right]\mathrm{d}t, \end{aligned} \tag{8}$$

where $i = 1, 2, \cdots, n$, $\left(\omega^{12}\dfrac{\partial B}{\partial a^2}\right)_i$, $\left(\omega^{21}\dfrac{\partial B}{\partial a^1}\right)_i$ and $\left(\dfrac{\partial \omega^{12}}{\partial a^1}\dfrac{\partial B}{\partial a^2} - \dfrac{\partial \omega^{12}}{\partial a^2}\dfrac{\partial B}{\partial a^1}\right)_i$ express the results obtained by using the ith parametric equations of heteroclinic orbit $a_i^1(t)$ and $a_i^2(t)$ instead of a^1 and a^2, respectively.

Especially, when $\dfrac{\partial \omega^{12}}{\partial a^1}\dfrac{\partial B}{\partial a^2} - \dfrac{\partial \omega^{12}}{\partial a^2}\dfrac{\partial B}{\partial a^1} \equiv 0$ (for example, when ω^{12}=1, the Birkhoff system becomes a Hamiltonian system), the Melnikov functions have simple forms as follows

$$\begin{aligned} M_i(t_0) = \int_{-\infty}^{+\infty} & \left[\left(\omega^{12}\frac{\partial B}{\partial a^2}\right)_i g_2(a_i^1(t), a_i^2(t), \lambda, t + t_0) \right. \\ & \left. - \left(\omega^{21}\frac{\partial B}{\partial a^1}\right)_i g_1(a_i^1(t), a_i^2(t), \lambda, t + t_0)\right]\mathrm{d}t. \end{aligned} \tag{9}$$

It can be proven that if $M_i(t_0)$ has simple zeros, for ε small enough, the unstable manifolds $W^u(p_{i,\varepsilon}^{t_0})$ (or the stable manifolds $W^s(p_{i,\varepsilon}^{t_0})$) and the stable manifolds $W^s(p_{i+1,\varepsilon}^{t_0})$ (or the unstable manifolds $W^u(p_{i+1,\varepsilon}^{t_0})$) must intersect transversally, that is, some transversal heteroclinic points exist. If for each i, $M_i(t_0)$ has simple zeros, then the Poincaré mapping $P_\varepsilon^{t_0}$ has n-transverse rings for ε small enough. So we can obtain:

Proposition 2. If the Melnikov functions $M_i(t_0)$ ($i = 1, 2, \cdots, n$) have simple zeros not depending on ε at the same time, then for ε small enough, the Poincaré mapping $P_\varepsilon^{t_0}$ of Eq.(6) has n-transverse rings, such that $P_\varepsilon^{t_0}$ has chaos in the sense of Smale horseshoe.

4. An illustrated example

We consider a second-order autonomous Birkhoff system as follows

$$R_1 = -\frac{1}{2}(a^2)^2, \quad R_2 = 0,$$

$$B = a^1(a^2)^2(f - ca^1 - da^2 - ea^1a^2), \qquad (10)$$

where c, d, e, f are all positive constants. Then the equations of motion of the system are

$$\dot{a}^1 = a^1(2f - 2ca^1 - 3da^2 - 3ea^1a^2),$$

$$\dot{a}^2 = a^2(-f + 2ca^1 + da^2 + 2ea^1a^2). \qquad (11)$$

Suppose that the following periodic perturbing model holds

$$\begin{aligned}\dot{a}^1 =& a^1(2f - 2ca^1 - 3da^2 - 3ea^1a^2)\\ &+ \varepsilon[2\lambda_1 a^1 + (\lambda_3 + \lambda_2 a^1)\cos\omega t],\\ \dot{a}^2 =& a^2(-f + 2ca^1 + da^2 + 2ea^1a^2)\\ &+ \varepsilon[-\lambda_1 a^2 + (\lambda_3 - \lambda_2 a^2)\cos\omega t]. \qquad (12)\end{aligned}$$

They are a predator–prey system[17] acted by a periodic perturbation $\varepsilon\lambda_3\cos\omega t$, and $2f + \varepsilon(2\lambda_1 + \lambda_2\cos\omega t)$ and $f + \varepsilon(\lambda_1 + \lambda_2\cos\omega t)$ express the intrinsic growth rates of the predator and prey groups respectively. All problems of this section are in the range of $G = \{(a^1, a^2)|a^1 \geq 0, a^2 \geq 0\}$.

System $(12)_{\varepsilon=0}$ has four singular points $O(0, 0)$, $B(f/c, 0)$, $C(0, f/d)$ and $D(a_0^1, a_0^2)$ in the range of G, where

$$a_0^1 = [-(4cd + ef) + \sqrt{(4cd + cf)^2 + 8cdef}]/(4ce),$$

$$a_0^2 = [(ef - 4cd) + \sqrt{(4cd - cf)^2 + 8cdef}]/(6ce), \qquad (13)$$

and O, B, C are hyperbolic saddle points, D is a centre. The integral curve of $(12)_{\varepsilon=0}$ is

$$B(a^1, a^2) = a^1(a^2)^2(f - ca^1 - da^2 - ea^1a^2) = h. \qquad (14)$$

When h=0, the curves corresponding to Eq.(14) are the rectilinear solutions a^1=0 and a^2=0, and the curvilinear solution $ca^1 + da^2 + ea^1a^2 = f$, which forms the heteroclinic orbits linking three saddle points O, B and C. The parametric equations of the heteroclinic orbits are, respectively,

$$OB : a^1(t) = \frac{f}{c(1 + \mathrm{e}^{-2ft})}, \quad a^2(t) = 0, \qquad (15)$$

$$OC : a^1(t) = 0, \quad a^2(t) = \frac{f}{d(1 + \mathrm{e}^{ft})}, \qquad (16)$$

$$BC : a^1(t) = \frac{f}{c(1 + \mathrm{e}^{ft})}, \quad a^2(t) = \frac{f}{d(1 + \mathrm{e}^{-ft})}. \qquad (17)$$

Now we begin to find the Melnikov functions of the three heteroclinic orbits of the system (12). The Melnikov function of the heteroclinic orbit OB is

$$\begin{aligned}M_1(t_0) =& \lambda_3 \int_{-\infty}^{+\infty} \exp\Big[-\int_0^t (f - 2ca^1(t))\mathrm{d}t\Big]\\ &\times (a^1(t))' \cos\omega(t + t_0)\mathrm{d}t,\end{aligned}$$

where $a^1(t)$ is determined by Eq.(15). So we can obtain

$$M_1(t_0) = \frac{\lambda_3 f}{2c\mathrm{ch}(\pi\omega/2f)}\cos\omega t_0.$$

The Melnikov function of the heteroclinic orbit OC is

$$\begin{aligned}M_2(t_0) =& -\lambda_3 \int_{-\infty}^{+\infty} \exp\Big[-\int_0^t (f - da^2(t))\mathrm{d}t\Big]\\ &\times (a^2(t))' \cos\omega(t + t_0)\mathrm{d}t,\end{aligned}$$

where $a^2(t)$ is determined by Eq.(16). So we can obtain

$$M_2(t_0) = \frac{\lambda_3\pi}{df\mathrm{sh}(\pi\omega/f)}[f^2\cos\omega t_0 - \omega^2\sin\omega t_0].$$

The Melnikov function of the heteroclinic orbit BC is

$$\begin{aligned}M_3(t_0) =& \int_{-\infty}^{+\infty} \exp\Big[-\int_0^t (f - 2ca^1(t) - da^2(t)\\ &- 2ea^1(t)a^2(t))\mathrm{d}t\Big]\{(a^1(t))'[-\lambda_1 a^2(t)\\ &+ (\lambda_3 - \lambda_2 a^2(t))\cos\omega(t + t_0)] - (a^2(t))'\\ &\times [2\lambda_1 a^1(t) + (\lambda_3 + \lambda_2 a^1(t))\cos\omega(t + t_0)]\}\mathrm{d}t,\end{aligned}$$

where $a^1(t)$ and $a^2(t)$ are determined by Eq.(17). So we can obtain

$$\begin{aligned}M_3(t_0) =& \frac{\pi\omega f}{3cd\mathrm{sh}(\pi\omega/f)}\\ &\times \{[3\lambda_3(c + d) - \lambda_2 f + 2\lambda_2 f\omega^2]\cos\omega t_0\\ &+ 3\omega[\lambda_3(c + d) - \lambda_2 f]\sin\omega t_0\}.\end{aligned}$$

It follows that, if

$$\lambda_3(c + d) - \lambda_2 f = 0,$$

then

$$3\lambda_3(c + d) - \lambda_2 f + 2\lambda_2 f\omega^2 = 2\lambda_3(c + d)(1 + \omega^2).$$

So, if $\lambda_3 \neq 0$, then $M_3(t_0)$ has simple zeros, and at the same time, it is easy to know that $M_1(t_0)$ and $M_2(t_0)$ have simple zeros too. So we judge that the system (12) has chaos in the sense of Smale horseshoes.

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