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Construction of the solution of variational equations for constrained Birkhoffian systems*

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In this paper we present the variational equations of constrained Birkhoffian systems and study their solution. It is proven that, under some conditions, a particular solution of variational equations can be obtained by using a first integral. At the end of the paper, an example is given to illustrate the application of the results.

Keywords: analytical mechanics, constrained Birkhoffian system, variational equation, first integral
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1. Introduction

For a holonomic conservative system, Whittaker[1] studied the problem involved in seeking a particular solution by using a given first integral. In 1991, Mei[2] extended the result to nonholonomic systems, and proved that, under some conditions, a solution of the variational equations of nonholonomic systems can be obtained by a given first integral.

A Birkhoffian system is a generalization of a Hamiltonian system,[3] and all holonomic constrained systems and nonholonomic constrained systems can be attributed to Birkhoffian systems.[4] Therefore, the research on dynamics of Birkhoffian systems has become an important branch in mathematics and physics, especially in modern analytical mechanics, and many important results have been obtained.[3−12]

In this paper, we consider a constrained Birkhoffian system. Firstly, the variational equations of the constrained Birkhoffian system are presented. Secondly, the solution of the variational equation is discussed, and it is proven that, under some conditions, a solution of the variational equations of constrained Birkhoffian systems can be obtained by a given first integral. Finally, an example is given to illustrate the application of the results.

2. Variational equations of the constrained Birkhoffian system

The Pfaff–Birkhoff–D'Alembert principle is expressed as[4]

$$\left\{\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}\right\}\delta a^\mu=0$$

$$(\mu,\nu=1,\cdots,2n). \tag{1}$$

Suppose that the variables a^μ of the system are not independent of each other, but are restricted by some constraints, for which the system is called the constrained Birkhoffian system. If the restrictions can be expressed as the following constraint equations

$$f_\beta(a^\mu,t)=0 \quad (\beta=1,\cdots,g;\mu=1,\cdots,2n). \tag{2}$$

The restrictions added to virtual displacements δa^μ by constraints (2) are

$$\frac{\partial f_\beta}{\partial a^\mu}\delta a^\mu=0 \quad (\beta=1,\cdots,g). \tag{3}$$

By principle (1) and Eqs.(3), using the method of Lagrange's multiplier, we can easily obtain the equations of motion with multipliers of the constrained Birkhoffian system

$$\Omega_{\mu\nu}\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=\lambda_\beta\frac{\partial f_\beta}{\partial a^\mu}$$

$$(\beta=1,\cdots,g;\mu,\nu=1,\cdots,2n), \tag{4}$$

where

$$\Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu} \tag{5}$$

is the Birkhoff covariant tensor. The motion of the constrained Birkhoffian system is determined by

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Eqs.(4) and (2). Eliminating $\dot{a}^\rho$ from Eqs.(2) and (4), we can obtain the algebraic equations with respect to λ_β

$$\frac{\partial f_\gamma}{\partial a^\rho}\Omega^{\rho\mu}\frac{\partial f_\beta}{\partial a^\mu}\lambda_\beta = -\frac{\partial f_\gamma}{\partial a^\rho}\Omega^{\rho\mu}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}\right)-\frac{\partial f_\gamma}{\partial t}$$
$$(\beta,\gamma=1,\cdots,g;\ \rho,\mu=1,\cdots,2n), \tag{6}$$

where $\Omega^{\rho\mu}$ is the Birkhoff contravariant tensor, and we have

$$\Omega^{\mu\nu}+\Omega^{\nu\mu}=0, \tag{7}$$

$$\Omega^{\tau\rho}\frac{\partial \Omega^{\mu\nu}}{\partial a^\rho}+\Omega^{\mu\rho}\frac{\partial \Omega^{\nu\tau}}{\partial a^\rho}+\Omega^{\nu\rho}\frac{\partial \Omega^{\tau\mu}}{\partial a^\rho}=0. \tag{8}$$

Let

$$\Delta=\det\left(\frac{\partial f_\gamma}{\partial a^\rho}\Omega^{\rho\mu}\frac{\partial f_\beta}{\partial a^\mu}\right), \tag{9}$$

when g is an odd number, there is Δ=0. Hence, we have

Theorem 1.[6] If the number of constraint equations of a constrained Birkhoffian system is odd, then the constraint multipliers λ_β have no unique solution.

When $\Delta\neq 0$, from Eqs.(6), we can obtain

$$\lambda_\beta=-\frac{\Delta_{\gamma\beta}}{\Delta}\left[\frac{\partial f_\gamma}{\partial a^\rho}\Omega^{\rho\mu}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}\right)+\frac{\partial f_\gamma}{\partial t}\right]$$
$$(\beta,\gamma=1,\cdots,g;\ \rho,\mu=1,\cdots,2n), \tag{10}$$

where $\Delta_{\gamma\beta}$ is an algebraic complement of element (γ,β) in the determinant Δ.

Let

$$P_\mu=P_\mu(t,\boldsymbol{a})=\lambda_\beta\frac{\partial f_\beta}{\partial a^\mu}\quad(\mu=1,\cdots,2n), \tag{11}$$

where λ_β is expressed as a function of t, $\boldsymbol{a}$ according to expression (10). Substituting Eqs.(11) into Eqs.(4), we obtain

$$\Omega_{\mu\nu}\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=P_\mu\quad(\mu,\nu=1,\cdots,2n). \tag{12}$$

The solutions of the constrained Birkhoffian systems (4) and (2) can be found from the solutions of Eqs.(12), if only the restriction of constraints (2) on the initial conditions is exerted, such as

$$f_\beta(a_0^\mu,t_0)=0\quad(\beta=1,\cdots,g). \tag{13}$$

We suppose that the system is non-singular, that is

$$\det(\Omega_{\mu\nu})\neq 0, \tag{14}$$

then from Eqs.(12) we can obtain

$$\dot{a}^\mu=\Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}+P_\nu\right)\quad(\mu=1,\cdots,2n). \tag{15}$$

In Eqs.(15), replacing a^μ with $a^\mu+\delta a^\mu$ and expanding the formulae with neglecting second-order and higher order infinitesimals, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta a^\mu=\Omega^{\mu\nu}\left[\frac{\partial^2 B}{\partial a^\nu\partial a^\rho}+\frac{\partial^2 R_\nu}{\partial t\partial a^\rho}+\frac{\partial P_\nu}{\partial a^\rho}-\frac{\partial \Omega_{\nu\chi}}{\partial a^\rho}\Omega^{\chi\sigma}\left(\frac{\partial B}{\partial a^\sigma}+\frac{\partial R_\sigma}{\partial t}+P_\sigma\right)\right]\delta a^\rho. \tag{16}$$

Equations (16) can be called the variational equations for constrained Birkhoffian systems.

3. First integrals with solutions of the variational equations of constrained Birkhoffian systems

We now study the connection of a given first integral with a solution of variational equations (16). First, we give a group of important relations about first integrals. Assuming that the system has a first integral in the form

$$\phi(t,a^\mu)=\text{const.}, \tag{17}$$

by differentiating Eq.(17) with respect to t, and using Eqs.(15), we obtain

$$\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial a^\nu}\Omega^{\nu\mu}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}+P_\mu\right)=0$$
$$(\nu,\mu=1,\cdots,2n). \tag{18}$$

Hence, we have

$$\frac{\partial}{\partial a^\rho}\left[\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial a^\nu}\Omega^{\nu\mu}\left(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t}+P_\mu\right)\right]=0$$
$$(\rho,\nu,\mu=1,\cdots,2n). \tag{19}$$

Using Eqs.(19) and (15), we can easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\phi}{\partial a^\rho}=-\frac{\partial\phi}{\partial a^\nu}\frac{\partial}{\partial a^\rho}\left[\Omega^{\nu\mu}\left(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}+P_\nu\right)\right]$$
$$(\rho,\mu,\nu=1,\cdots,2n). \tag{20}$$

Formulae (20) are a group of important relations about first integrals.

Secondly, we seek the solution of the variational equations (16). Assuming Eqs.(16) have a solution in the form

$$\delta a^\mu=\varepsilon\Omega^{\mu\nu}\frac{\partial\phi}{\partial a^\nu}+\varepsilon A^\mu(t,a^\nu), \tag{21}$$

where ε is a small parameter. Substituting Eqs.(21) into Eqs.(16), and using Eqs.(20) and relations (7) and (8), we can obtain

$$\begin{aligned}\frac{\mathrm{d}A^\mu}{\mathrm{d}t} =&\Omega^{\mu\nu}\left[\frac{\partial^2 B}{\partial a^\nu \partial a^\rho}+\frac{\partial^2 R_\nu}{\partial t\partial a^\rho}+\frac{\partial P_\nu}{\partial a^\rho}\right.\\ &\left.-\frac{\partial \Omega_{\nu\chi}}{\partial a^\rho}\Omega^{\chi\sigma}\left(\frac{\partial B}{\partial a^\sigma}+\frac{\partial R_\sigma}{\partial t}+P_\sigma\right)\right]A^\rho\\ &+\Omega^{\mu\rho}\left(\frac{\partial P_\rho}{\partial a^\omega}-\frac{\partial P_\omega}{\partial a^\rho}\right)\Omega^{\omega\kappa}\frac{\partial \phi}{\partial a^\kappa}. \end{aligned} \tag{22}$$

Equations (22) are the equations to determine $A^\mu(\mu = 1,\cdots,2n)$. If one can find A^μ from Eqs.(22), one can obtain the solution of variational equations (16) in the form of Eqs.(21). Of course, generally speaking, it is very difficult to find the general solution of Eqs.(22).

Finally, we study the relation between a first integral and a particular solution of the variational equations. If $P_\mu(\mu = 1,\cdots,2n)$ and the first integral ϕ satisfy the following conditions

$$\Omega^{\mu\rho}\left(\frac{\partial P_\rho}{\partial a^\omega}-\frac{\partial P_\omega}{\partial a^\rho}\right)\Omega^{\omega\kappa}\frac{\partial \phi}{\partial a^\kappa}=0 \quad (\mu=1,\cdots,2n), \tag{23}$$

then Eqs.(22) become

$$\begin{aligned}\frac{\mathrm{d}A^\mu}{\mathrm{d}t} =&\Omega^{\mu\nu}\left\{\frac{\partial^2 B}{\partial a^\nu \partial a^\rho}+\frac{\partial^2 R_\nu}{\partial t\partial a^\rho}+\frac{\partial P_\nu}{\partial a^\rho}\right.\\ &\left.-\frac{\partial \Omega_{\nu\chi}}{\partial a^\rho}\Omega^{\chi\sigma}\left(\frac{\partial B}{\partial a^\sigma}+\frac{\partial R_\sigma}{\partial t}+P_\sigma\right)\right\}A^\rho. \end{aligned} \tag{24}$$

Obviously, Eqs.(24) have a particular solution

$$A^\mu = 0 \quad (\mu = 1,\cdots,2n), \tag{25}$$

and Eqs.(21) have the form

$$\delta a^\mu = \varepsilon\Omega^{\mu\nu}\frac{\partial \phi}{\partial a^\nu} \quad (\mu = 1,\cdots,2n). \tag{26}$$

Hence, we have the following theorem.

Theorem 2. If the first integral ϕ of the constrained Birkhoffian systems (4) and (2) satisfies conditions (23), then the particular solutions of the variational equations (16) of the system can be given by Eqs.(26) which are constructed through the first integral ϕ.

4. An illustrative example

Example[7] Consider a fourth-order Birkhoffian system, the Birkhoffian functions of the system are

$$B=\frac{1}{2}[(a^1)^2+(a^3)^2+(a^4)^2], \tag{27}$$

$$R_1 = a^3, \quad R_2 = a^4, \quad R_3 = R_4 = 0, \tag{28}$$

and the constraint equations are

$$f_1 = a^1a^3 - c_1^2 = 0, \quad f_2 = a^1 + a^4 - c_2 = 0. \tag{29}$$

Now we study the first integral with the construction of the particular solution of variational equations for the system.

First, we establish the variational equations of the system. From Eqs.(5), we can easily obtain

$$\Omega_{\mu\nu}=\begin{pmatrix}0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&1&0&0\end{pmatrix}, \tag{30}$$

and

$$\Omega^{\mu\nu}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}. \tag{31}$$

Equations (4) give

$$\begin{aligned}&-\dot{a}^3 - a^1 = \lambda_1 a^3 + \lambda_2, \quad -\dot{a}^4 = 0,\\ &\dot{a}^1 - a^3 = \lambda_1 a^1, \quad \dot{a}^2 - a^4 = \lambda_2. \end{aligned} \tag{32}$$

From constraint equations (29) and (32), we have

$$\lambda_1 = -\frac{a^3}{a^1}, \quad \lambda_2 = -a^1 + \frac{(a^3)^2}{a^1}. \tag{33}$$

Therefore, we obtain

$$\begin{aligned}&P_1 = \lambda_1 a^3 + \lambda_2 = -a^1, \quad P_2 = 0,\\ &P_3 = \lambda_1 a^1 = -a^3, \quad P_4 = \lambda_2 = -a^1 + \frac{(a^3)^2}{a^1}. \end{aligned} \tag{34}$$

Equations (15) give

$$\begin{aligned}&\dot{a}^1 = 0, \quad \dot{a}^2 = a^4 - a^1 + \frac{(a^3)^2}{a^1},\\ &\dot{a}^3 = 0, \quad \dot{a}^4 = 0. \end{aligned} \tag{35}$$

The variational equations (16) of the system give

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\delta a^1 = 0,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\delta a^2 = \delta a^4 - \left(1+\frac{(a^3)^2}{(a^1)^2}\right)\delta a^1 + \frac{2a^3}{a^1}\delta a^3,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\delta a^3 = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}\delta a^4 = 0. \end{aligned} \tag{36}$$

It is easy to verify that the system has the following first integrals[7]

$$\phi_1 = a^4 = \text{const.}, \tag{37}$$

$$\phi_2 = a^2 - a^4 t + a^1 t - \frac{(a^3)^2}{a^1} t = \text{const.} \qquad (38)$$

Substituting formula (31), Eqs.(34) and integral (37) into the condition (23), we know that the condition (23) can be satisfied, and according to theorem 2 of this paper, the variational equations (36) of the system have a particular solution which corresponds with the first integral (37), that is

$$\delta a^1 = 0, \quad \delta a^2 = \varepsilon, \quad \delta a^3 = 0, \quad \delta a^4 = 0. \qquad (39)$$

We can easily verify that the first integral (38) does not satisfy the condition (23), and therefore the particular solution of the variational equations (36), which is in the form of formula (38), does not exist.

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