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Theory of symmetry for a rotational relativistic Birkhoff system*

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The theory of symmetry for a rotational relativistic Birkhoff system is studied. In terms of the invariance of the rotational relativistic Pfaff–Birkhoff–D'Alembert principle under infinitesimal transformations, the Noether symmetries and conserved quantities of a rotational relativistic Birkhoff system are given. In terms of the invariance of rotational relativistic Birkhoff equations under infinitesimal transformations, the Lie symmetries and conserved quantities of the rotational relativistic Birkhoff system are given.

Keywords: rotational relativity, Birkhoff system, Noether symmetry, Lie symmetry
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1. Introduction

In 1979, Bengtsson and Frauendorf[1] accurately measured the maximum values of the spin velocity of fourteen kinds of nucleon, and the results showed that those maximum values were different from each other. With the development of science and technology, more and more experimental facts have been related to high-speed spinning. In 1986, Carmeli[2] constructed the theory of rotational relativistic mechanics, and provided the theoretical basis for studying the problems of high-speed spinning. In recent years, we have given the Hamilton principle, D'Alembert principle and many kinds of differential equations of motion of rotational relativistic systems. Furthermore, we have studied the algebraic structure, integral methods, Noether symmetries and Lie symmetries of this system, and constructed the theory of analytical mechanics for a rotational relativistic system.[3−7]

In 1927, the American mathematician Birkhoff made primary researches on Birkhoff dynamics.[8] In 1983, the American physicist Santilli studied the transformation theory of Birkhoff equations and the generalization of Galilei relativity, and summarized comprehensively the origin of Birkhoff equations and the later studies on them.[9] Later, the Chinese scientist Mei Feng-xiang gave the Birkhoff equations of a nonholonomic constrained system, and studied the Noether theory, stability of motion and geometrical description of a Birkhoff system, as well as constructing the theoretical frame of Birkhoff dynamics.[10−20] The Birkhoff dynamics is more general than the Hamilton dynamics. The Hamilton dynamics has been extensively applied to the field of modern physics, so the Birkhoff dynamics will also be applied to this field.

Recently, we have given the basic theory, integral method and geometrical theory of a rotational relativistic Birkhoff system in Refs.[21]–[23]. In this paper, the theory of symmetry for a rotational relativistic Birkhoff system is studied. In terms of the invariance of the Pfaff–Birkhoff–D'Alembert principle under infinitesimal transformations, the theorem of Noether symmetries and its inverse theorem for a rotational relativistic Birkhoff system are constructed. In terms of the invariance of Birkhoff equations under infinitesimal transformations, the theorem of Lie symmetries and its inverse theorem for a rotational relativistic Birkhoff system are also constructed.

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2. Pfaff–Birkhoff–D'Alembert principle and Birkhoff equations for a rotational relativistic system

We study the fixed-axis rotation of a mechanical system consisting of N particles. At time t the angular velocity of the ith particle is $\dot{\theta}_i$, the limiting angular velocity is Γ_i, the classical rotational moment of inertia is I_o, and its relativistic rotational moment of inertia is[2,3]

$$I_i = \frac{I_{oi}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}}. \tag{1}$$

Constructing the Birkhoffian B^* and the Birkhoff functions R_ν^* (ν=1, $\cdots$, $2n$) of the rotational relativistic system

$$B^* = B^*(I_i(t,a^\mu),t,a^\mu),$$
$$R_\nu^* = R_\nu^*(I_i(t,a^\mu),t,a^\mu), \tag{2}$$

and letting

$$\tilde{B}^* = \tilde{B}^*(t,a^\mu) = B^*(I_i(t,a^\mu),t,a^\mu),$$
$$\tilde{R}_\nu^* = \tilde{R}_\nu^*(t,a^\mu) = R_\nu^*(I_i(t,a^\mu),t,a^\mu), \tag{3}$$

then the Pfaff–Birkhoff–D'Alembert principle of a rotational relativistic system can be written in the form[21,22]

$$\left[\left(\frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial\tilde{B}^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial t}\right]\delta a^\mu = 0$$
$$(\nu,\mu=1,\cdots,2n), \tag{4}$$

where the repeated subscripts represent taking summation.

For an ideal holonomic or free rotational relativistic Birkhoff system, the δa^μ (μ=1, $\cdots$, $2n$) in Eq.(4) are independent of each other, so we obtain

$$\left(\frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}\right)\dot{a}^\nu-\frac{\partial\tilde{B}^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial t}=0$$
$$(\nu,\mu=1,\cdots,2n). \tag{5}$$

Equations (5) are called the Birkhoff equations of a rotational relativistic system.[21,22] In general, suppose the system (5) is non-singular, i.e.

$$\det(\tilde{\omega}_{\mu\nu}^*)\neq 0, \quad \tilde{\omega}_{\mu\nu}^* = \left(\frac{\partial\tilde{R}_\nu^*}{\partial a^\mu}-\frac{\partial\tilde{R}_\mu^*}{\partial a^\nu}\right). \tag{6}$$

If $\tilde{R}_\mu^*$ and $\tilde{B}^*$ do not include time t explicitly, the system (5) is autonomous. If $\tilde{R}_\mu^*$ does not include time t explicitly, the system (5) is semi-autonomous.

Example 1. Now we study the high-speed spinning of a body around a fixed axis, its classical rotational moment of inertia is I_0, and the relativistic equations of motion are[2,3]

$$\ddot{q} = \frac{M_0}{I}\left(1-\frac{\dot{q}^2}{\Gamma^2}\right) = \frac{M_0}{I_0}\left(1-\frac{\dot{q}^2}{\Gamma^2}\right)^{3/2},$$
$$I = \frac{I_0}{\sqrt{1-\dot{q}^2/\Gamma^2}}. \tag{7}$$

Letting $a^1 = q$ and $a^2 = \dot{q}$, then Eqs.(7) can be expressed as the first-order standard form

$$\dot{a}^1 = a^2, \quad \dot{a}^2 = \frac{M_0}{I_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{3/2}. \tag{8}$$

It is easy to solve Eqs.(8) and to obtain two first integrals independent of each other

$$\Phi_1 = \frac{I_0}{M_0}\Gamma\mathrm{tg}\left(\arcsin\frac{a^2}{\Gamma}\right)-t, \quad \Phi_2 = \frac{I}{M_0}\Gamma^2-a^1. \tag{9}$$

Using the Hojman method to solve $\tilde{B}^*$ and $\tilde{R}_\nu^*$,[9] we have

$$\tilde{B}^* = G_1, \quad \tilde{R}_1^* = -G_2,$$
$$\tilde{R}_2^* = G_1\frac{I_0}{M_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{-3/2} + G_2\frac{I_0}{M_0}a^2\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{-3/2}. \tag{10}$$

Choosing $G_1=\Phi_2$, G_0=0, with $\det\left(\frac{\partial G_\mu}{\partial I_\nu}-\frac{\partial G_\nu}{\partial I_\mu}\right)\neq 0$ being satisfied, we obtain

$$\tilde{B}^* = \frac{I}{M_0}\Gamma^2-a^1, \quad R_1^*=0,$$
$$\tilde{R}_2^* = \frac{I}{M_0}\left(\frac{1}{M_0}\Gamma^2-a^1\right)\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{-1}, \tag{11}$$

in which $\tilde{B}^*$, $\tilde{R}_1^*$ and $\tilde{R}_2^*$ do not include time t apparently, hence they satisfy the autonomous Birkhoff equations for rotational relativistic system. Equations (11) describe the motion of the rotational relativistic system.

3. Noether symmetries and conserved quantities of a rotational relativistic Birkhoff system

In 1918, German mathematician Noether studied the invariance of Hamilton action under infinitesimal transformations, and revealed the relation between the conserved quantities of a physical system

and its symmetries.[24] The Noether theorem greatly promoted the development of modern mechanics and theoretical physics. In the past 30 years, Candottie, Djukić, Vujanović, Li Zi-ping, Bahar, Liu Duan, Mei Feng-xiang, Zhao Yue-yu and Luo Shao-kai have further promoted the study of Noether theory, and brought it into many branches.[10,15,20,24–40] Most studies on the Noether theorem are about the invariance of Hamilton action under infinitesimal transformations. In 1993, Mei Feng-xiang constructed a new Noether theory in terms of the invariance of Pfaff action under infinitesimal transformations.[10] Now we study the Noether symmetries and conserved quantities of a rotational relativistic Birkhoff system in terms of the variable form of the Pfaff–Birkhoff–D'Alembert principle under infinitesimal transformations, and give the corresponding theorem of Noether symmetries and its inverse theorem.

3.1. Noether theorem of a rotational relativistic Birkhoff system

We introduce infinitesimal transformations for t and a^μ

$$t^* = t + \Delta t, \quad a^{*\mu}(t^*) = a^\mu(t) + \Delta a^\mu, \tag{12}$$

and let

$$\Delta t = \varepsilon f(t, a^\mu), \quad \Delta a^\mu = \varepsilon F_\mu(t, a^\mu), \tag{13}$$

where f and F_μ are generators of time and space respectively under the infinitesimal transformations (12), and ε is a small parameter. Isochronous variation δa^μ and non-isochronous variation Δa^μ satisfy

$$\Delta a^\mu = \delta a^\mu + \dot{a}^\mu \Delta t, \tag{14}$$

so we have

$$\delta a^\mu = \Delta a^\mu - \dot{a}^\mu \Delta t = \varepsilon(F_\mu - \dot{a}^\mu f). \tag{15}$$

Substituting Eq.(15) into Eq.(4), we obtain

$$\varepsilon\left\{\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial t}\right\}(F_\mu - \dot{a}^\mu f) = 0. \tag{16}$$

Noticing

$$\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu \dot{a}^\mu = 0,$$

$$\frac{\partial \tilde{B}^*}{\partial a^\mu}\dot{a}^\mu f = \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{B}^* \dot{f}) - \tilde{B}^* \dot{f} - \frac{\partial \tilde{B}^*}{\partial t} f, \tag{17}$$

$$\left(\frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\dot{a}^\nu + \frac{\partial \tilde{R}^*_\mu}{\partial t}\right)F_\mu = \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{R}^*_\mu F_\mu) - \tilde{R}^*_\mu \dot{F}_\mu,$$

we can rewrite Eq.(16) as

$$\varepsilon\left\{\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu}\right)F_\mu + \left(\frac{\partial \tilde{R}^*_\mu}{\partial t}\dot{a}^\mu + \frac{\partial \tilde{B}^*}{\partial t}\right)f + \tilde{R}^*_\mu \dot{F}_\mu - \tilde{B}^* \dot{f} - \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{R}^*_\mu F_\mu - \tilde{B}^* f)\right\} = 0. \tag{18}$$

Introducing a gauge function $P = P(t, a^\mu)$, and finding the sum for $\varepsilon \dot{P}$ in Eq.(14), we obtain the following transformation satisfying the invariance of principle (4)

$$\varepsilon\left\{\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu}\right)F_\mu + \tilde{R}^*_\mu \dot{F}_\mu + \left(\frac{\partial \tilde{R}^*_\mu}{\partial t}\dot{a}^\mu - \frac{\partial \tilde{B}^*}{\partial t}\right)f - \tilde{B}^* \dot{f} + \dot{P} - \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{R}^*_\mu F_\mu - \tilde{B}^* f + P)\right\} = 0. \tag{19}$$

Equation (19) is the variable form of the Pfaff–Birkhoff–D'Alembert principle of a rotational relativistic Birkhoff system under infinitesimal transformations.

From the invariance condition (19), we obtain the Noether theorem of a rotational relativistic Birkhoffian system.

Theorem 1. For the rotational relativistic Birkhoff system (5), if the generators f, F_μ and the gauge function P satisfy

$$\dot{P} + \left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu}\right)F_\mu + \left(\frac{\partial \tilde{R}^*_\mu}{\partial t}\dot{a}^\mu - \frac{\partial \tilde{B}^*}{\partial t}\right)f + \tilde{R}^*_\mu \dot{F}_\mu - \tilde{B}^* \dot{f} = 0 \qquad (\nu, \mu = 1, \cdots, 2n), \tag{20}$$

then the system possesses the following conserved quantity

$$\tilde{R}^*_\mu F_\mu - \tilde{B}^* f + P = \text{const.} \tag{21}$$

Equation (20) is called the Noether equation of the rotational relativistic Birkhoff system (5). For the autonomous and semi-autonomous systems, the Noether equation (20) becomes a simple form, and theorem 1 still holds. For the given $\tilde{B}^*$ and $\tilde{R}^*_\nu$, we can firstly find the infinitesimal transformation generators f, F_μ and the gauge function P, then obtain the conserved quantities from Eq.(21).

3.2. Generalized Killing equation of a rotational relativistic Birkhoff system

For obtaining the functions f, F_μ and P, expanding Eq.(20), we obtain

$$\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}\dot{a}^\nu F_\mu - \frac{\partial \tilde{B}^*}{\partial a^\mu}F_\mu + \tilde{R}^*_\mu \frac{\partial F_\mu}{\partial t} + \tilde{R}^*_\mu \frac{\partial F_\mu}{\partial a^\nu}\dot{a}^\nu + \frac{\partial \tilde{R}^*_\mu}{\partial t}\dot{a}^\mu f - \frac{\partial \tilde{B}^*}{\partial t}f - \tilde{B}^* \frac{\partial f}{\partial t} - \tilde{B}^* \frac{\partial f}{\partial a^\mu}\dot{a}^\mu + \frac{\partial P}{\partial t} + \frac{\partial P}{\partial a^\mu}\dot{a}^\mu = 0. \tag{22}$$

Separating the terms containing $\dot{a}^\nu$ and other terms not containing $\dot{a}^\nu$, and taking their coefficients as zeros, we have

$$\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}F_\mu+\frac{\partial F_\mu}{\partial a^\nu}\tilde{R}^*_\mu-\tilde{B}^*\frac{\partial f}{\partial a^\nu}+\frac{\partial \tilde{R}^*_\nu}{\partial t}f+\frac{\partial P}{\partial a^\nu}=0, \quad (23)$$

$$\tilde{R}^*_\mu\frac{\partial F_\mu}{\partial t}-\frac{\partial \tilde{B}^*}{\partial a^\mu}F_\mu-\tilde{B}^*\frac{\partial f}{\partial t}-\frac{\partial \tilde{B}^*}{\partial t}f+\frac{\partial P}{\partial t}=0. \quad (24)$$

Equations (23) and (24) are $2n+1$ partial differential equations of $2n+2$ functions f, F_μ and P, and are called the generalized Killing equations of the rotational relativistic Birkhoff system. If the equations have solutions, then we can obtain the functions f, F_μ and P. Hence, theorem 1 can also be expressed as

Theorem 2. For the rotational relativistic Birkhoff system (5), if the infinitesimal transformation generators f, F_μ and the gauge function P satisfy the generalized Killing equations (23) and (24), then the system possesses the conserved quantity as Eq.(21).

3.3. Noether inverse theorem of a rotational relativistic Birkhoff system

If the rotational relativistic Birkhoff system (5) has r first integrals independent of each other

$$\tilde{\Phi}^*=\tilde{\Phi}^*(t,a^\mu)=\Phi^*(I_i(t,a^\mu),t,a^\mu)=\text{const.}, \quad (25)$$

then we can obtain the corresponding Noether symmetries from those integrals.

For Eq.(25), we have

$$\frac{\mathrm{d}\tilde{\Phi}^*}{\mathrm{d}t}=0. \quad (26)$$

Multiplying Eq.(5) by $(F_\mu-\dot{a}^\mu f)$, and finding the sum for μ, then adding it to Eq.(26), we obtain

$$\begin{aligned}&\frac{\partial \tilde{\Phi}^*}{\partial t}+\frac{\partial \tilde{\Phi}^*}{\partial I_i}\frac{\partial I_i}{\partial t}+\frac{\partial \tilde{\Phi}^*}{\partial a^\mu}\dot{a}^\mu+\frac{\partial \tilde{\Phi}^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\dot{a}^\mu\\&+\left\{\left(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\right.\\&\left.-\left(\frac{\partial \tilde{B}^*}{\partial a^\mu}+\frac{\partial \tilde{R}^*_\mu}{\partial t}\right)\right\}(F_\mu-\dot{a}^\mu f)=0\\&\qquad(\nu,\mu=1,\cdots,2n;i=1,\cdots,N). \quad (27)\end{aligned}$$

Taking the coefficient of $\dot{a}^\nu$ in Eq.(27) as zero, we obtain

$$\begin{aligned}&\frac{\partial \tilde{\Phi}^*}{\partial a^\nu}+\frac{\partial \tilde{\Phi}^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}+\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}-\frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)F_\mu\\&+\left(\frac{\partial \tilde{B}^*}{\partial a^\nu}+\frac{\partial \tilde{R}^*_\nu}{\partial t}\right)f=0\\&\qquad(\nu,\mu=1,\cdots,2n;i=1,\cdots,N). \quad (28)\end{aligned}$$

If the rotational relativistic Birkhoff system (5) is nonsingular, i.e. $\det(\tilde{\omega}^*_{\mu\nu})\neq 0$, then from Eq.(22) we have

$$F_\mu=\tilde{\omega}^*_{\mu\nu}\left\{\frac{\partial \tilde{\Phi}^*}{\partial a^\nu}+\frac{\partial \tilde{\Phi}^*}{\partial I_i}\frac{\partial I_i}{\partial a^\nu}+\left(\frac{\partial \tilde{B}^*}{\partial a^\nu}+\frac{\partial \tilde{R}^*_\nu}{\partial t}\right)f\right\}, \quad (29)$$

where $\tilde{\omega}^*_{\mu\nu}\tilde{\omega}^*_{\nu\rho}=\delta_{\mu\rho}$. Letting integral (25) equal the conserved quantity (21), i.e.

$$\tilde{\Phi}^*=\Phi^*(I_i(t,a^\mu),t,a^\mu)=\tilde{R}^*_\mu F_\mu-\tilde{B}^*f+P$$

$$(\mu=1,\cdots,2n), \quad (30)$$

then the following Noether inverse theorem of the rotational relativistic Birkhoff system holds.

Theorem 3. If we have obtained the first integrals of the rotational relativistic Birkhoff system (5), chosen the gauge function P, and determined the infinitesimal transformation generators f and F_μ of the system from Eqs.(29) and (30), then they correspond to the Noether symmetrical transformations of the rotational relativistic Birkhoff system (5).

Example 2. It is known that the Birkhoffian and Birkhoff functions of a rotational relativistic system are Eqs.(11), then we try to study the Noether symmetries and conserved quantities of the system.

Substituting Eqs.(11) into Noether equation (20), we have

$$f=1,\quad F_1=0,\quad F_2=0,\quad P=0, \quad (31)$$

$$f=-1,\quad F_1=0,\quad F_2=0,\quad P=0, \quad (32)$$

$$f=0,\quad F_1=a^2,\quad F_2=\frac{M_0}{I}(1-(a^2)^2/I^2),\quad P=0, \quad (33)$$

$$f=0,\quad F_1=-a^2,$$

$$F_2=-\frac{M_0}{I}(1-(a^2)^2/I^2),\quad P=0. \quad (34)$$

Substituting Eqs.(31) and (34) into Eq.(21), and substituting Eqs.(32) and (33) into Eq.(21), we can obtain the conserved quantities

$$\Phi^*_1=a^1-\frac{I}{M_0}I^2=\text{const.}, \quad (35)$$

$$\Phi^*_2=\frac{I}{M_0}-a^1=\text{const.} \quad (36)$$

Equations (31), (34) and Eqs.(32), (33) are linear correlation, so the conserved quantities (35) and (36) are also linear correlation.

Example 3. It is known that the rotational relativistic Birkhoff system has integrals (35) and (36),

then we try to solve the symmetrical transformations corresponding to Eq.(29).

Because

$$(\tilde{\omega}^*_{\mu\nu}) = \frac{I_0}{M_0}(1-(a^2)^2/I^2)^{-3/2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (37)$$

then

$$(\tilde{\omega}^{*\mu\nu}) = \frac{M_0}{I_0}(1-(a^2)^2/I^2)^{3/2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (38)$$

For the integral Φ_1^*, Eqs.(29) and (30) give

$$F_1 = -a^2(1-f), \quad F_2 = -\frac{M_0}{I}(1-(a^2)^2/I^2)(1-f), \quad (39)$$

$$f = 0, \quad F_1 = -a^2,$$

$$F_2 = \frac{M_0}{I}(1-(a^2)^2/I^2), \quad P = 0. \quad (40)$$

From Eqs.(39) and (40), we can obtain Eqs.(31) and (34), which are infinitesimal symmetrical transformations corresponding to Φ_1^*. For integral Φ_2^*, Eqs.(29) and (30) give

$$F_1 = a^2(1+f), \quad F_2 = \frac{M_0}{I}(1-(a^2)^2/I^2), \quad P = 0, \quad (41)$$

$$\frac{I}{M_0}I^2 - a^1 = \left(\frac{I}{M_0}I^2 - a^1\right)\frac{I}{M_0}(1-(a^2)^2/I^2)^{-1}F_2 - \left(\frac{I}{M_0}I^2 - a^1\right)f + P. \quad (42)$$

From Eqs.(41) and (42), we can obtain Eqs.(32) and (33), which are infinitesimal symmetrical transformations corresponding to Φ_2^*.

4. Lie symmetries and conserved quantities of a rotational relativistic Birkhoff system

At the end of the 19th century, the well-known Norwegian mathematician Sophus Lie studied the invariance of differential equations under infinitesimal transformations, that is Lie symmetries. Olver[41], Bluman and Kumei[42] and Ibragimov[43] further promoted the studies of Lie theory by mathematical methods. In 1979, Lutzky[44] applied Lie theory to the differential equations of motion for mechanical systems, and studied Lie symmetries and the conserved quantity of a dynamical system. In the past 20 years, Prince, Thompson, Lakshmann, Mei Fengxiang, Zhao Yue-yu, Zhang Yi and Luo Shao-kai developed the method of Lie symmetries, and obtained many important results.[7,17,20,40,45–57] Now, in terms of the invariance of the rotational relativistic Birkhoff equations under infinitesimal transformations, we study the Lie symmetries and conserved quantities of the rotational relativistic Birkhoff system, construct the determining equation of Lie symmetries, obtain the structure equation and conserved quantities, and study the inverse problem of Lie symmetries of this system.

4.1. Lie symmetry theorem for a rotational relativistic Birkhoff system

Introducing infinitesimal transformations (12) for t and a^μ, and noticing Eqs.(13), we have

$$t^* = t + \varepsilon f(t, a^\mu), \quad a^{*\mu} = a^\mu + \varepsilon F_\mu(t, a^\mu). \quad (43)$$

We introduce a vector of infinitesimal transformation of a generator

$$X^{(0)} = f\frac{\partial}{\partial t} + F_\mu\frac{\partial}{\partial a^\mu}, \quad (44)$$

and its first extension

$$X^{(1)} = f\frac{\partial}{\partial t} + F_\mu\frac{\partial}{\partial a^\mu} + (\dot{F}_\mu - \dot{a}^\mu\dot{f})\frac{\partial}{\partial \dot{a}^\mu}. \quad (45)$$

From the invariance of the differential equation under infinitesimal transformations, we know that the invariance of Eqs.(5) under the infinitesimal transformations (43) is

$$X^{(1)}\left[\dot{a}^\mu - \tilde{\omega}^{*\mu\nu}\left(\frac{\partial \tilde{B}^*}{\partial a^\nu} + \frac{\partial \tilde{R}^*_\nu}{\partial t}\right)\right] = 0. \quad (46)$$

Noticing the inverse equations of Eqs.(5), we can obtain the determining equation of the rotational relativistic Birkhoff system

$$\dot{F}_\mu - \tilde{\omega}^{*\mu\nu}\left(\frac{\partial \tilde{B}^*}{\partial a^\nu} + \frac{\partial \tilde{R}^*_\nu}{\partial t}\right)\dot{f} = X^{(0)}\left[\tilde{\omega}^{*\mu\nu}\left(\frac{\partial \tilde{B}^*}{\partial a^\nu} + \frac{\partial \tilde{R}^*_\nu}{\partial t}\right)\right]$$

$$(\nu, \mu = 1, \cdots, 2n). \quad (47)$$

If the generators f and F_μ of the infinitesimal transformations (43) satisfy the determining equation (47), then the transformations (43) are called Lie symmetries. So we have the following Lie symmetry theorem for the rotational relativistic Birkhoff system.

Theorem 4. For the infinitesimal transformation generators f and F_μ satisfying the determining

equation (47), if there exists a gauge function P satisfying the following condition

$$X^{(1)}(\tilde{R}^*_\mu \dot{a}^\mu - \tilde{B}^*) + (\tilde{R}^*_\mu \dot{a}^\mu - \tilde{B}^*)\dot{f} + \dot{P} = 0, \quad (48)$$

then the rotational relativistic Birkhoff system has the conserved quantities as follows

$$\tilde{\Phi}^* = \tilde{R}^*_\mu F_\mu - \tilde{B}^* f + P = \text{const.} \quad (49)$$

In fact

$$\begin{aligned}\frac{\mathrm{d}\tilde{\Phi}^*}{\mathrm{d}t} =& \dot{\tilde{R}}^*_\mu F_\mu + \tilde{R}^*_\mu \dot{F}_\mu - \dot{\tilde{B}}^* f - \tilde{B}^* \dot{f} \\ & - X^{(1)}(\tilde{R}^*_\mu \dot{a}^\mu - \tilde{B}^*) - (\tilde{R}^*_\mu \dot{a}^\mu - \tilde{B}^*)\dot{f} \\ =& \left[\left(\frac{\partial R^*_\mu}{\partial a^\nu} + \frac{\partial R^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial a^\nu} - \frac{\partial R^*_\nu}{\partial a^\mu} - \frac{\partial R^*_\nu}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\dot{a}^\nu \right. \\ & \left. + \left(\frac{\partial B^*}{\partial t} + \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial t} + \frac{\partial B^*}{\partial a^\mu} + \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu}\right)\right] F_\mu \\ & - \left(\frac{\partial B^*}{\partial a^\mu} + \frac{\partial B^*}{\partial I_i}\frac{\partial I_i}{\partial a^\mu} + \frac{\partial R^*_\mu}{\partial t} + \frac{\partial R^*_\mu}{\partial I_i}\frac{\partial I_i}{\partial t}\right)\dot{a}^\nu f \\ =& \left[\left(\frac{\partial \tilde{R}^*_\mu}{\partial a^\nu} - \frac{\partial \tilde{R}^*_\nu}{\partial a^\mu}\right)\dot{a}^\nu + \frac{\partial \tilde{B}^*}{\partial a^\mu} + \frac{\partial \tilde{R}^*_\mu}{\partial t}\right] F_\mu \\ & - \left(\frac{\partial \tilde{B}^*}{\partial a^\mu} + \frac{\partial \tilde{R}^*_\mu}{\partial t}\right)\dot{a}^\nu f,\end{aligned}$$

using Eqs.(14), we have

$$\left(\frac{\partial \tilde{B}^*}{\partial a^\mu} + \frac{\partial \tilde{R}^*_\mu}{\partial t}\right)\dot{a}^\mu = \left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu \dot{a}^\mu = 0.$$

So $\mathrm{d}\tilde{\Phi}^*/\mathrm{d}t$=0, i.e. Φ^*=const.

To sum up, the steps of studying Lie symmetries of the rotational relativistic Birkhoff system are as follows. Firstly, we substitute the given rotational relativistic Birkhoff equations into Eq.(47), and obtain the generators f and F_μ. Secondly, substituting f and F_μ into the structure equation (48), and obtaining $\dot{P}$, if $\dot{P}$=0 or $\dot{P}$ is a perfect differential of a function, then we can obtain a gauge function. Finally, substituting f, F_μ and P into Eq.(49), we can obtain the conserved quantities.

4.2. Inverse theorem of Lie symmetries of a rotational relativistic Birkhoff system

If the rotational relativistic Birkhoff system has r first integrals (25) independent of each other, then we can obtain the corresponding Lie symmetry transformations from these integrals.

Similar to the discussion about the inverse problem of Noether symmetries from Eqs.(26) to (30), if the first integrals are known, and the gauge function P is chosen, then the generators for the infinitesimal transformations, f and F_μ, can be obtained from Eqs.(29) and (30), which correspond to the Noether symmetrical transformations of the rotational relativistic Birkhoff system.

If the generators f and F_μ satisfy the determining equation (47), then the transformations are Lie symmetrical transformations of the system, otherwise they are not Lie symmetrical transformations of the system. So we have the following inverse theorem of Lie symmetries of the rotational relativistic Birkhoff system.

Theorem 5. If we have known r first integrals independent of each other of the rotational relativistic Birkhoff system (14), we can obtain generators f and F_μ of the infinitesimal transformations from Eqs.(29) and (30), and if f and F_μ satisfy the determining equation (47), then the transformations are Lie symmetrical transformations corresponding to the system (5) and the integral (25), otherwise they are not Lie symmetrical transformations.

Example 4. The Birkhoffian and Birkhoff functions of the rotational relativistic second-order Birkhoff system are Eqs.(11), and we try to study Lie symmetries and conserved quantities of the system.

Substituting Eqs.(11) and (38) into the determining equation (47), we obtain

$$\dot{F}_1 - a^2 \dot{f} = F_2, \quad (50)$$

$$\dot{F}_2 - \frac{M_0}{I_0}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{3/2}\dot{f} = -\frac{3M_0 a^2}{I_0 \Gamma^2}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{1/2} F_2, \quad (51)$$

they have solutions

$$f = 1, \quad F_1 = 0, \quad F_2 = 0, \quad (52)$$

$$f = 0, \quad F_1 = a^2, \quad F_2 = \frac{M_0}{I_0}\left(1 - \frac{(a^2)^2}{\Gamma^2}\right)^{3/2}. \quad (53)$$

Substituting Eqs.(52) and (53) into Eq.(49), we have the conserved quantity as follows

$$\tilde{\Phi}^* = \frac{I}{M_0}\Gamma^2 - a^1 = \text{const.} \quad (54)$$

Example 5. The Birkhoffian and Birkhoff functions of a rotational relativistic second-order Birkhoff system are Eqs.(29), and it is known that the system has the integral (54). We try to study the corresponding Lie symmetrical transformations.

$\tilde{\omega}^*_{\mu\nu}$ and $\tilde{\omega}^{*\mu\nu}$ can be obtained respectively from Eqs.(37) and (38). For the integral (54), we can obtain from Eqs.(29) and (30)

$$F_1 = -a^2(1 - f), \quad (55)$$

$$F_2 = -\frac{M_0}{I_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{3/2}(1-f), \tag{56}$$

$$\begin{aligned} &F_2\left(\frac{I}{M_0}\Gamma^2-a^1\right)\frac{I}{M_0}\left(1-\frac{(a^2)^2}{\Gamma^2}\right)^{-1} \\ &-\left(\frac{I}{M_0}\Gamma^2-a^1\right)f+P \\ =&\frac{I}{M_0}\Gamma^2-a^1. \end{aligned} \tag{57}$$

Choosing $P=0$, we can obtain the solutions similar to Eqs.(52) and (53) from Eqs.(55)–(57), which satisfy the determining equations (47), and are Lie symmetrical transformations corresponding to the known conserved quantity.

5. Discussion

We have proven that the canonical equations of the rotational relativistic Hamilton system are a special example of Eqs.(5) in Ref.[21]. In this paper, if R^*_μ does not include time t, for the rotational relativistic Birkhoff system (5), and we let

$$a^\mu = \begin{cases} q^\mu & (\mu=1,\cdots,n), \\ p_{\mu-n} & (\mu=n+1,\cdots,2n), \end{cases}$$

$$R^*_\mu = \begin{cases} p_\mu & (\mu=1,\cdots,n), \\ 0 & (\mu=n+1,\cdots,2n), \end{cases}$$

$$B^* = H_r, \tag{58}$$

then the Birkhoff tensors of the rotational relativistic system are

$$(\tilde{\omega}^*_{\mu\nu}) = \begin{bmatrix} 0_{n\times n} & -1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{bmatrix}. \tag{59}$$

We can obtain immediately the canonical equations of the rotational relativistic Hamilton system

$$\dot{q}^s = \frac{\partial H_r}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H_r}{\partial q^s} \quad (s=1,\cdots,n), \tag{60}$$

where

$$H_r(t,q^s,p_s) = \dot{p}^s\dot{q}_s - L_r, \quad p_s = \frac{\partial L_r}{\partial \dot{q}_s},$$

$$L_r(t,q^s,\dot{q}^s) = T^*_r - V, \quad T^*_r = I_{oi}\Gamma_i^2\left(1-\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}\right)$$
$$(s=1,\cdots,n; i=1,\cdots,N). \tag{61}$$

Therefore, in this paper we give the theory of symmetry of the rotational relativistic Hamilton system.

Under the classical approximation $\dot{\theta}_i \ll \Gamma_i$,

$$I_i = \frac{I_{oi}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}} \approx I_{oi} \quad (i=1,\cdots,N), \tag{62}$$

$$\begin{aligned} \tilde{B}^* &= B^*(I_i(t,a^\mu),t,a^\mu) \approx B(t,a^\mu), \\ \tilde{R}^*_\nu &= R^*_\nu(I_i(t,a^\mu),t,a^\mu) \approx R_\nu(t,a^\mu). \end{aligned} \tag{63}$$

Therefore, in this paper we give the theory of symmetry of the classical rotational Birkhoff system.

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