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Bäcklund transformation and multiple soliton solutions for the (3+1)-dimensional Jimbo–Miwa equation*

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We study an approach to constructing multiple soliton solutions of the (3+1)-dimensional nonlinear evolution equation. We take the (3+1)-dimensional Jimbo–Miwa (JM) equation as an example. Using the extended homogeneous balance method, one can find a Bäcklund transformation to decompose the (3+1)-dimensional JM equation into a linear partial differential equation and two bilinear partial differential equations. Starting from these linear and bilinear partial differential equations, some multiple soliton solutions for the (3+1)-dimensional JM equation are obtained by introducing a class of formal solutions.

Keywords: soliton, homogeneous balance method, (3+1)-dimensions, JM equation
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1. Introduction

Solving nonlinear problems plays an important and significant part in nonlinear science. It is well known that there are infinitely many solutions for every nonlinear evolution equation and it is difficult to seek an exact solution of a real nonlinear evolution equation. Much attention is being paid to the study of the nonlinear evolution equation by researchers in mathematics, physics, biology and engineering fields. In order to find the exact and explicit solutions, particularly solitary wave solutions, many important and effective methods have been developed, such as the inverse scattering approach, Hirota's bilinear method, Lie symmetry method, Bäcklund transformation, Darboux transformation, Ansatz method, and so on. Recently, Wang *et al.*[1] presented first a powerful homogeneous balance method for finding exact solutions of a given nonlinear evolution equation. This method provides a convenient analytical technique to construct exact solutions[2] and has been generalized to obtain multi-soliton solutions,[3] dromion solution,[4] and auto-Bäcklund transformation.[4,5] Recently, Fan[6] further discussed the connection between the homogeneous balance method, the Weiss–Tabor–Carnevale method and the Clarkson–Kruskal method. In this paper, we explore some interesting multi-soliton solutions for the (3+1)-dimensional Jimbo–Miwa (JM) equation

$$u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} = 0 \qquad (1)$$

by means of the extended homogeneous balance method. It has been shown that the (3+1)-dimensional JM equation (1), which was introduced by Jimbo and Miwa,[7] does not have the Painlevé property[8] as defined by Weiss *et al.*[9]

The paper is organized as follows. In Sect.2, a general theory of the extended homogeneous balance method is described. In Sect.3, a Bäcklund transformation for Eq.(1), which is simplified into a set of linear and bilinear partial differential equations, is obtained by using the extended homogeneous balance method. In Sect.4, the abundant multiple soliton solutions for Eq.(1) are constructed. Section 5 includes a simple discussion.

2. General theory of extended homogeneous balance method

Here we first describe the general theory of extended homogeneous balance method. For a given n-

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dimensional Nth-order partial differential equation

$$P(x_0 \equiv t, x_1, x_2, \cdots, x_n, u, u_{x_i}, u_{x_i x_j}, \cdots, u_{x_{i_1} x_{i_2} \cdots x_{i_n}}) = 0, \quad (2)$$

where the function P is a polynomial function of the field u and its derivatives. A function $\phi = \phi(x_0, x_1, x_2, \cdots, x_n)$ is called a quasi-solution of Eq.(1), if there exists a function $f = f(\phi)$ with one variable only, so that a suitable linear combination of

$$1, f(\phi), f(\phi)_{x_i}, f(\phi)_{x_i x_j}, \cdots \quad (3)$$

is actually a solution of Eq.(2). The homogeneous balance method mainly consists of the following five steps:

(i) As the leading part of analysis, select a suitable linear combination of the functions in Eq.(3), its coefficients to be determined, such that the highest nonlinear terms and the highest order partial derivative terms in the given equation (2) can be partially balanced.

(ii) Substituting the chosen combination into Eq.(1), the given equation (2) is transformed into polynomials with the same highest degree in partial derivatives of $\phi(x_0, x_1, \cdots, x_n)$ in spite of $f(\phi)$ and its derivatives of different degrees. Collecting all terms with the highest degree of derivatives of $\phi(x_0, x_1, \cdots, x_n)$ and setting its coefficient to zero, an ordinary differential equation of $f(\phi)$ is obtained.

(iii) Starting from the ordinary differential equation of $f(\phi)$, not only the solution of $f(\phi)$ is obtained, but also the nonlinear terms of various derivatives of $f(\phi)$ in the expression obtained in the second step can be replaced by the corresponding higher order derivatives of $f(\phi)$.

(iv) After some detailed calculations, collect terms with the same order of derivatives of $f(\phi)$ and setting the coefficient of each order of derivative of $f(\phi)$ to zero, respectively, a set of linear and bilinear, even trilinear, partial differential equations for function $\phi(x_0, x_1, \cdots, x_n)$ are obtained.

(v) By solving the equations obtained in the fourth step, we can find many solutions of $\phi(x_0, x_1, \cdots, x_n)$.

(vi) Making use of the results of $\phi(x_0, x_1, \cdots, x_n)$, abundant exact solution structures of given equation (2) can be constructed.

3. Bäcklund transformation of the JM equation

According to the extended homogeneous balance method, let

$$u = f_x(\phi) + u_1(x, y, z, t) = f'\phi_x + u_1(x, y, z, t), \quad (4)$$

where u_1 is a solution of Eq.(1). Substituting Eq.(4) into Eq.(1), we obtain

$$\begin{aligned}
&u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} \\
=&(6f''f^{(3)} + f^{(5)})\phi_y\phi_x^4 + f^{(4)}(4\phi_x^3\phi_{xy} \\
&+ 6\phi_y\phi_x^2\phi_{xx}) + 6f''^2(\phi_x^3\phi_{xy} + 2\phi_y\phi_x^2\phi_{xx}) \\
&+ 3f'f^{(3)}(\phi_x^3\phi_{xy} + \phi_y\phi_x^2\phi_{xx}) \\
&+ f^{(3)}(2\phi_t\phi_x\phi_y - 3\phi_z\phi_x^2 + 3\phi_y u_{1x}\phi_x^2 \\
&+ 3u_{1y}\phi_x^3 + 12\phi_x\phi_{xy}\phi_{xx} + 3\phi_y\phi_{xx}^2 \\
&+ 6\phi_x^2\phi_{xxy} + 4\phi_y\phi_x\phi_{xxx}) \\
&+ f'f''(15\phi_x\phi_{xy}\phi_{xx} + 3\phi_y\phi_{xx}^2 + 3\phi_x^2\phi_{xxy} \\
&+ 3\phi_y\phi_x\phi_{xxx}) + f''(2\phi_{yt}\phi_x + 2\phi_y\phi_{xt} \\
&- 6\phi_x\phi_{xz} + 3\phi_x^2 u_{1xy} + 2\phi_t\phi_{xy} + 6u_{1x}\phi_x\phi_{xy} \\
&+ 3\phi_y\phi_x u_{1xx} - 3\phi_z\phi_{xx} + 3\phi_y u_{1x}\phi_{xx} \\
&+ 9u_{1y}\phi_x\phi_{xx} + 6\phi_{xx}\phi_{xxy} + 4\phi_{xy}\phi_{xxx} \\
&+ 4\phi_x\phi_{xxxy} + \phi_y\phi_{xxxx}) + f'^2(3\phi_{xx}\phi_{xxy} \\
&+ 3\phi_{xy}\phi_{xxx}) + f'(2\phi_{xyt} + 3\phi_{xy}u_{1xx} \\
&+ 3u_{1xy}\phi_{xx} - 3\phi_{xxz} + 3u_{1x}\phi_{xxy} \\
&+ 3u_{1y}\phi_{xxx} + \phi_{xxxxy}) = 0. \quad (5)
\end{aligned}$$

Setting the coefficients of $\phi_y\phi_x^4$ in Eq.(5) to zero yields an ordinary differential system for $f(\phi)$

$$6f''f^{(3)} + f^{(5)} = 0. \quad (6)$$

The following special solutions exist for Eq.(6):

$$f(\phi) = 2\ln(\phi). \quad (7)$$

Thereby $f''^2 = -\frac{1}{3}f^{(4)}$, $f'f^{(3)} = -\frac{2}{3}f^{(4)}$, $f'f'' = -f^{(3)}$, $f'^2 = -2f''$. Using these results, expression (5) can be simplified as

$$\begin{aligned}
&u_{xxxy} + 3u_{xy}u_x + 3u_y u_{xx} + 2u_{yt} - 3u_{xz} \\
=&(2\phi_t\phi_x\phi_y - 3\phi_z\phi_x^2 - 3\phi_y\phi_x^2 u_{1x} 3u_{1y}\phi_x^3 \\
&- 3\phi_x\phi_{xy}\phi_{xx} + 3\phi_x^2\phi_{xxy} + \phi_y\phi_x\phi_{xxx})f^{(3)} \\
&+ (2\phi_{yt}\phi_x + 2\phi_y\phi_{xt} - 6\phi_x\phi_{xz} + 3\phi_x^2 u_{1xy} \\
&+ 2\phi_t\phi_{xy} + 6u_{1x}\phi_x\phi_{xy} + 3\phi_y\phi_x u_{1xx} \\
&- 3\phi_z\phi_{xx} + 3\phi_y u_{1x}\phi_{xx} + 9u_{1y}\phi_x\phi_{xx} \\
&- 2\phi_{xy}\phi_{xxx} + 4\phi_x\phi_{xxxy} + \phi_y\phi_{xxxx})f'' \\
&+ (2\phi_{xyt} + 3\phi_{xy}u_{1xx} + 3u_{1xy}\phi_{xx} + 3\phi_{xxz} \\
&+ 3u_{1x}\phi_{xxy} + 3u_{1y}\phi_{xxx} + \phi_{xxxxy})f' = 0. \quad (8)
\end{aligned}$$

Putting the coefficients of $f^{(3)}$, f'', f' in Eq.(8) to zero yields a set of partial differential equations for $\phi(x,y,z,t)$:

$$2\phi_t\phi_x\phi_y - 3\phi_z\phi_x^2 - 3\phi_y\phi_x^2 u_{1x} + 3u_{1y}\phi_x^3 - 3\phi_x\phi_{xy}\phi_{xx} + 3\phi_x^2\phi_{xxy} + \phi_y\phi_x\phi_{xxx} = 0, \quad (9)$$

$$2\phi_{yt}\phi_x + 2\phi_y\phi_{xt} - 6\phi_x\phi_{xz} + 3\phi_x^2 u_{1xy} + 2\phi_t\phi_{xy} + 6u_{1x}\phi_x\phi_{xy} + 3\phi_y\phi_x u_{1xx} - 3\phi_z\phi_{xx} + 3\phi_y u_{1x}\phi_{xx} + 9u_{1y}\phi_x\phi_{xx} - 2\phi_{xy}\phi_{xxx} + 4\phi_x\phi_{xxxy} + \phi_y\phi_{xxxx} = 0, \quad (10)$$

$$2\phi_{xyt} + 3\phi_{xy}u_{1xx} + 3u_{1xy}\phi_{xx} - 3\phi_{xxz} + 3u_{1x}\phi_{xxy} + 3u_{1y}\phi_{xxx} + \phi_{xxxxy} = 0. \quad (11)$$

Analysing Eqs.(9)–(11), if u_1=0, we find

$$2\phi_{yt} - 3\phi_{xz} + \phi_{xxxy} = 0, \quad (12)$$

$$2\phi_t\phi_y - 3\phi_x\phi_z + \phi_y\phi_{xxx} = 0, \quad (13)$$

$$\phi_x\phi_{xxy} - \phi_{xy}\phi_{xx} = 0. \quad (14)$$

Then Eqs.(9)–(11) are satisfied. Thereby, from Eqs.(4) and (7), we can get a class of Bäcklund transformation for Eq.(1) as follows

$$u = \frac{\partial}{\partial x}\ln\phi, \quad (15)$$

where ϕ satisfies Eqs.(12)–(14).

4. Multiple soliton solutions of the JM equations

Now we can construct the multiple soliton-like solution of Eq.(1) by means of Eqs.(12)–(14) . From Eqs.(12)–(14), two special solutions read as

$$\phi_1 = 1 + \sum_{i=1}^{n}\exp(k_i x + ly + hz + \omega_i t) \quad (16)$$

and

$$\phi_2 = 1 + \sum_{i=1}^{n}\exp(kx + ly + h_i z + \omega_i' t), \quad (17)$$

where

$$\omega_i = \frac{3k_i h - lk_i^3}{2l}, \quad \omega_i' = \frac{3kh_i - lk^3}{2l}. \quad (18)$$

Inserting Eqs.(16) and (17) into Eq.(15), respectively, we can construct two types of multiple soliton-like solutions of the JM equation:

$$u_1 = \frac{2\sum_{i=1}^{n} k_i\exp(k_i x + ly + hz + \omega_i t)}{1 + \sum_{i=1}^{n}\exp(k_i x + ly + hz + \omega_i t)} \quad (19)$$

and

$$u_2 = \frac{2\sum_{i=1}^{n} k\exp(kx + ly + h_i z + \omega_i' t)}{1 + \sum_{i=1}^{n}\exp(kx + ly + h_i z + \omega_i' t)}. \quad (20)$$

It is clearly seen that as in the (2+1)-dimensional case, the field u is not exponentially localized in all directions. In the (2+1)-dimensional case, although the field u is not exponentially localized in all directions, one of its potentials decays in all directions. Similarly, considering the properties of some potentials of the field u, we can also construct the corresponding multiple dromion solution of the potential form of Eq.(1), that is

$$u_{1x} = -\frac{2\sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j\exp[(k_i + k_j)x + 2ly + 2hz + (\omega_i + \omega_j)t]}{\left[1 + \sum_{i=1}^{n}\exp(k_i x + ly + hz + \omega_i t)\right]^2} + \frac{2\sum_{i=1}^{n} k_i^2\exp(k_i x + ly + hz + \omega_i t)}{1 + \sum_{i=1}^{n}\exp(k_i x + ly + hz + \omega_i t)}, \quad (21)$$

$$u_{2x} = -\frac{2k^2\sum_{i=1}^{n}\sum_{j=1}^{n}\exp[2kx + 2ly + (h_i + h_j)z + (\omega_i' + \omega_j')t]}{\left[1 + \sum_{i=1}^{n}\exp(kx + ly + h_i z + \omega_i' t)\right]^2} + \frac{2k^2\sum_{i=1}^{n}\exp(kx + ly + h_i z + \omega_i' t)}{1 + \sum_{i=1}^{n}\exp(kx + ly + h_i z + \omega_i' t)}, \quad (22)$$

$$u_{1y} = -\frac{2l\sum_{i=1}^{n}\sum_{j=1}^{n} k_i \exp[(k_i+k_j)x+2ly+2hz+(\omega_i+\omega_j)t]}{\left[1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_it)\right]^2} + \frac{2l\sum_{i=1}^{n} k_i\exp(k_ix+ly+hz+\omega_it)}{1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_it)}, \quad (23)$$

$$u_{2y} = -\frac{2kl\sum_{i=1}^{n}\sum_{j=1}^{n} \exp[2kx+2ly+(h_i+h_j)z+(\omega_i'+\omega_j')t]}{\left[1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_i't)\right]^2} + \frac{2kl\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_i't)}{1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_i't)}, \quad (24)$$

$$u_{1z} = -\frac{2h\sum_{i=1}^{n}\sum_{j=1}^{n} k_i \exp[(k_i+k_j)x+2ly+2hz+(\omega_i+\omega_j)t]}{\left[1+\sum_{i=1}^{n}\exp(k_ix+ly+hz+\omega_it)\right]^2} + \frac{2h\sum_{i=1}^{n} k_i\exp(k_ix+ly+hz+\omega_it)}{1+\sum_{i=1}^{n}\exp(k_ix+ly+hz+\omega_it)}, \quad (25)$$

$$u_{2z} = -\frac{2k\sum_{i=1}^{n}\sum_{j=1}^{n} h_j \exp[2kx+2ly+(h_i+h_j)z+(\omega_i'+\omega_j')t]}{\left[1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_i't)\right]^2} + \frac{2k\sum_{i=1}^{n} h_i\exp(kx+ly+h_iz+\omega_i't)}{1+\sum_{i=1}^{n}\exp(kx+ly+h_iz+\omega_i't)}. \quad (26)$$

If we take n=1 and n=2, different results similar to that given in Ref.[4] can be constructed.

5. Conclusion

In summary, by using the extended homogeneous balance method for the (3+1)-dimensional JM equation (1), some types of single, multi-soliton solutions, multiple dromion solutions are obtained. The method used here is standard and direct, so we believe that these solutions may exist for other classes of (3+1)-dimensional nonlinear mathematical physics models. Further work on these aspects is worthy of study.

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