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On the problem of reality conditions for the Ashtekar formulation*

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In this paper we discuss the reality conditions for Lorentzian and Euclidean gravity in the Ashtekar formulation by introducing a double conformal transformation. We generalize Marugan's results and demonstrate that the values of the double conformal factor have to be either real or double complex numbers. Either Lorentzian or Euclidean gravitational theory is up to the different values of the double conformal factor. Furthermore, the reality conditions of Lorentzian and Euclidean gravitational theory can be expressed in a unified way by use of the double complex function method.

Keywords: Ashtekar formulation; reality conditions; conformal transformation
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1. Introduction

At present, there is some consensus about the reasons behind the success of the Ashtekar formulations.[1−3] One reason is the geometrical nature of the Ashtekar variables. In particular, the fact that the configuration variable is a connection is especially interesting because this allows us to use loop variables at both the classical and quantum levels.[4,5] Another advantage of the Ashtekar theory is the simplicity of the constraints, especially the Hamiltonian constraint, that has been very helpful in finding solutions to all of them.

In spite of all the success of the formulation, there are still several problems that we are mostly concerned with in this paper, one of which is the issue of the reality conditions.[1−3,6−11] As is well known, the so-called reality conditions must be imposed on the complex Ashtekar variables in order to recover the usual real formulation of general relativity for space-times with Lorentzian signatures. Their role is to guarantee that both the 3-metric and its time derivative (evolution under the action of the Hamiltonian constraints) are real. But this introduces key difficulties to the canonical quantization of gravity.[12−14]

By use of the conformal transformation, Marugan[17] successfully derived the reality conditions for Lorentzian and Euclidean gravity in the Ashtekar formulation. But he only discussed the reality conditions by starting from Lorentzian gravity, and he found that there is a limitation for the conformal factor Ω.

By generalizing Marugan's method, we introduce a double conformal transformation, through which we analyse the reality conditions for Lorentzian and Euclidean gravity. Compared with Marugan's method[17], we can find more solutions of the conformal factor Ω by the double complex function method (DCFM).[15,16] Hence we can unitedly express the reality conditions for the Lorentzian and Euclidean gravitational theories.

We find that under the double conformal transformation, if one requires that the classical real triad is represented by a Hermitian operator in these theories, the compatibility of the *-operation with the algebra of commutator and constraints restricts the admissible conformal factor to be either real number or hyperbolic complex unit (i.e. hyperbolic pure imaginary unit $\epsilon, \epsilon^2 = +1, \epsilon \neq 1$).[16] One recovers the Lorentzian sector of general relativity and the well-known set of reality conditions in the case of the double complex unit. For the real number and the elliptic complex unit (i.e. ordinary pure imaginary unit $i, i^2 = -1$), the classical theory can be identified as real Euclidean gravity, and the corresponding set of reality conditions guarantee that any real observable derived from the Euclidean 3-metric and extrinsic curvature is self-adjoint in the quantum theory. There-

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fore, these reality conditions can be interpreted as associated with Euclidean general relativity. Explicitly, the Euclidean Ashtekar connection must be Hermitian, and the densitized triad has to be represented by an anti-Hermitian operator.

Penrose's abstract index notation is used throughout. Space-time and spatial tensor indices are denoted by Latin letters from the beginning of the alphabet a, b, c..., while internal indices are denoted by Latin indices from the middle of the alphabet i, j, k....

2. Ashtekar gravitational theory

In the Lorentzian gravity of the Ashtekar theory, the Ashtekar variables can be defined as a densitized triad, $\tilde{E}^a{}_i$, and a SO(3) connection, $A_a{}^i$ over a 3-manifold Σ, satisfying $\tilde{E}^a{}_i = \mathrm{e}^a{}_i q(e), A_a{}^i = \Gamma_a{}^i(e) - iK_{ab}\mathrm{e}^{bi}$.[1-3] The canonical conjugate variables are the arbitrary complex fields $(\tilde{E}^a{}_i, A_a{}^i)$, with the Poisson brackets

$$\begin{aligned}\{A_a{}^i(x), A_b{}^j(y)\} =&0,\\ \{A_a{}^i(x), \tilde{E}^b{}_j(y)\} =&i\delta^b_a\delta^i_j\delta^3(x,y),\\ \{\tilde{E}^a{}_i(x), \tilde{E}^b{}_j(y)\} =&0,\end{aligned} \tag{1}$$

where the SO(3) indices are raised and lowered with the metric $\eta^{ij} = (1,1,1)$, $q^{ab} = \mathrm{e}^a{}_i\mathrm{e}^{bi}$, is the inverse of 3-metric, $q(e) = (\det q_{ab})^{\frac{1}{2}}$, and $\Gamma_a{}^i(e)$ is SO(3) connection compatible with the triad

$$\Gamma_a{}^i = \frac{1}{2}\varepsilon^{ijk}E_{jb}(-\partial_a\tilde{E}^b{}_k + \Gamma^b{}_{ca}\tilde{E}^c{}_k). \tag{2}$$

Here, ε^{ijk} is the anti-symmetric symbol, E_{jb} denotes the inverse of the densitized triad, and $\Gamma^a{}_{bc}$ are the Christoffel symbols[15]

$$\Gamma^a{}_{bc} = \frac{1}{2}q^{ad}(\partial_c q_{db} + \partial_b q_{dc} - \partial_d q_{bc}). \tag{3}$$

Finally, the extrinsic curvature can be expressed in terms of the lapse function N, the shift functions N^a, and the time derivatives of the 3-metric[15]

$$K_{ab} = \frac{1}{2N}(\partial_t q_{ab} - D_{(a}N_{b)}), \tag{4}$$

with $D_{(a}N_{b)}$ the symmetrized covariant derivative of $N_a = q_{ab}N^b$.

In the Ashtekar formulation, the constraints for Lorentzian gravity can be written as

$$\begin{aligned}\tilde{G}_i \equiv&D_a\tilde{E}^a{}_i = 0,\\ \tilde{V}_a \equiv&F_{ab}{}^i\tilde{E}^b{}_i = 0,\\ \tilde{\tilde{S}} \equiv&\varepsilon^{ijk}\tilde{E}^a{}_i\tilde{E}^b{}_jF_{abk} = 0.\end{aligned} \tag{5}$$

$F_{ab}{}^i$ is the curvature of the SO(3) connection $A_a{}^i$.

The constraints in Eq.(5) are usually called the Gauss constraint, the vector constraint and the scalar constraint (i.e. the so-called Hamiltonian constraint), respectively.

In the Ashtekar formulation, the Hamiltonian can be expressed as

$$H = \int_\Sigma d^3x\left(\frac{1}{2}N\tilde{\tilde{S}} - iN^a(\tilde{V}_a - A_a{}^i\tilde{G}_i) + iN^i\tilde{G}_i\right). \tag{6}$$

Here $\tilde{G}_i, \tilde{V}_a, \tilde{\tilde{S}}$ are the constraints given in Eq.(5), $\overset{N}{\sim} = Nq^{-\frac{1}{2}}$ is the densitized lapse function, N^a is the shift function and N^i is a SO(3) multiplier.

The Hamiltonian (6) can lead to the equations of motion[2,3]

$$\begin{aligned}\dot{\tilde{E}}^a{}_i =&\{\tilde{E}^a{}_i, H\}\\ =&-iD_b(\overset{N}{\sim}\epsilon_i{}^{jk}\tilde{E}^a{}_j\tilde{E}^b{}_k)\\ &+\pounds_{\vec{N}}(\tilde{E}^a{}_i) + N^k\tilde{E}^a{}_j\epsilon_i{}^j{}_k,\\ \dot{A}^i_a =&\{A_a{}^i, H\}\\ =&i\overset{N}{\sim}\epsilon^{ij}{}_k\tilde{E}^b{}_jF_{ab}{}^k + \pounds_{\vec{N}}(A_a{}^i) + D_aN^i,\end{aligned} \tag{7}$$

where the dot denotes time derivative, $\pounds_{\vec{N}}$ is the Lie derivative

$$\begin{aligned}\pounds_{\vec{N}}(\tilde{E}^a{}_i) =&\partial_b(N^b\tilde{E}^a{}_i) - \tilde{E}^b{}_i\partial_bN^a,\\ \pounds_{\vec{N}}(A_a{}^i) =&N^b\partial_bA_a{}^i + A_b{}^i\partial_aN^b,\end{aligned} \tag{8}$$

and D_b is the derivative operator defined by the Ashtekar connection

$$\begin{aligned}D_b(N\epsilon_i{}^{jk}\tilde{E}^a{}_j\tilde{E}^b{}_k) =&\partial_b(N\epsilon_i{}^{jk}\tilde{E}^a{}_j\tilde{E}^b{}_k)\\ &+\epsilon_{il}{}^mA_b{}^lN\epsilon_m{}^{jk}\tilde{E}^a{}_j\tilde{E}^b{}_k,\\ D_aN^i =&\partial_aN^i + \epsilon^i{}_{jk}A_a{}^jN^k.\end{aligned} \tag{9}$$

As discussed above, in the Ashtekar formulation there is an important problem to be solved, i.e. the so-called reality conditions. We know that self-dual connection in the Ashtekar theory are complex variables, hence it is a complex theory. However, as a gravitational theory, it must degenerate into Einstein's general relativity under some special conditions. So one imposes the reality conditions on the Ashtekar complex variable in order to recover the usual real formulation of general relativity for space-times with Lorentzian signatures. But Ashtekar gave the denotation of the reality conditions just in form,[3] i.e., $(\tilde{E}^a{}_i\tilde{E}^{bi})$ is real and positive, and its derivative with respect to time is also real. In fact, the reality conditions given by Ashtekar are not valid in the canonical quantization program.[17] Therefore, at present the

problem has become an important research subject of the Ashtekar's theory.

3. Doubly Conformally Lorentzian and Euclidean gravity

For the convenience of the following discussion, here we briefly introduce the DCFM (more details have been presented in Refs.[16] and [18]–[20]).

Let J denote the double imaginary unit. $J = i$ is the ordinary pure imaginary unit, satisfying $i^2 = -1$; $J = \epsilon$ is the hyperbolic complex unit, satisfying $\epsilon^2 = 1$ and $\epsilon \neq 1$. $Z_{(J)} = a_{(J)} + Jb_{(J)}$ is called a double complex number, where $a_{(J)}$ and $b_{(J)}$ are both double real numbers.Then when $J = i$, $Z_{(J=i)} = Z_c = a_c + ib_c$ is the elliptic (ordinary) complex number; when $J = \epsilon$, $Z_{(J=\epsilon)} = Z_H = a_H + \epsilon b_H$ is the hyperbolic complex number. Unlike the elliptic complex, all of the hyperbolic complex numbers $a_H + \epsilon b_H$ (a_H, b_H are real numbers) comprise a ring, not a field.

In order to generalize Marugan's method, now we introduce the double conformal transformation on a space-time manifold M_P with the Lorentzian or Euclidean signature, i.e., the space-time line element satisfies

$$\mathrm{d}s^2 = \Omega^2(J)(\mathrm{d}s_P)^2. \tag{10}$$

Here, $\Omega(J)$ is a double-exponent function whose absolute value is equal to unity, i.e. $\Omega = e^{J\theta}, \theta \in [0, 2\pi)$.

In the space-time manifold M_P, the 3-metric $(q_P)_{ab}$ determined by $(\mathrm{d}s_P)^2$ will be assumed to be real and non-negative. The lapse and shift functions associated with $(\mathrm{d}s_P)^2$ will be denoted by N_P and N_P^a, respectively.

Under the double conformal transformation defined by Eq.(10), the lapse and shift functions and the 3-metric are related by

$$N = \Omega(J)N_P, \quad N^a = N_P^a, \quad q_{ab} = \Omega^2(J)(q_P)_{ab}. \tag{11}$$

So the transformation law for the triad is

$$\mathrm{e}^a{}_i = \Omega^{-1}(J)(e_P)^a{}_i. \tag{12}$$

Using Eqs.(11) and (12), we can derive the relations between the densitized triads and the SO(3) connections of the two considered four-metrics

$$\tilde{E}^a{}_i = \Omega^2(J)(\tilde{E}_P)^a{}_i, \ A_a{}^i = (A_P)_a{}^i. \tag{13}$$

Substituting the equations in the gravitational constraints (5), we can obtain

$$\begin{aligned} \tilde{G}_i \equiv& \Omega^2(J)(\tilde{G}_P)_i = 0, \\ \tilde{V}_a \equiv& \Omega^2(J)(\tilde{V}_P)_a = 0, \\ \tilde{S} \equiv& \Omega^4(J)\tilde{\tilde{S}}_P = 0, \end{aligned} \tag{14}$$

with the subscript P denoting the evaluation at $(\tilde{E}_P)^a{}_i$ and $(A_P)_a{}^i$. Therefore, the gravitational constraints for the Ashtekar variables $\tilde{E}^a{}_i$ and $A_a{}^i$ can be equivalently written by simply evaluating them at the conformally transformed variables.

From the definition of $\overset{N}{\sim}$ and Eq.(11), we also conclude that

$$N = \Omega^{-2}(J)N_P, \ N^a = N_P^a. \tag{15}$$

It is then possible to see that the dynamical equations (7) are still valid when evaluated at the Lagrange multipliers and the Ashtekar variables determined. The Poisson brackets between $(\tilde{E}_P)^a{}_i$ and $(A_P)_a{}^i$, together with the Hamiltonian, satisfy

$$\begin{aligned} &\{(A_P)_a{}^i(x), (\tilde{E}_P)^b{}_j(y)\} \\ =& -i\Omega^{-2}(J)\delta_a^b\delta_j^i\delta^3(x,y), \\ H =& \Omega^2(J)H_P. \end{aligned} \tag{16}$$

4. Doubly conformally reality conditions for the Ashtekar formulation

Firstly, for later use, let us define a *-relation (i.e. an involution) on this algebra. If the two elementary classical variables F and G are each complex conjugates, i.e. if $\overline{F} = G$, then the corresponding operators $\hat{F}$ and $\hat{G}$ satisfy $\hat{G} = (\hat{F})^*$, and have the following properties of an involution[3]

$$\begin{aligned} (\hat{F} + \lambda\hat{G})^* =& \hat{F}^* + (\overline{\lambda})\hat{G}^*, \\ (\hat{F}\hat{G})^* =& \hat{G}^*\hat{F}^*, \\ (\hat{F}^*)^* =& F. \end{aligned} \tag{17}$$

Here, λ is a complex number, and "–" denotes the complex conjugate.

Under the double conformal transformation, we can derive the reality conditions for Lorentzian and Euclidean gravity through the compatibility of the *-operation with the algebra of commutator and constraints.

In order to extend Marugan's theory we begin our discussion from the more general case, i.e. from Lorentzian and Euclidean gravity simultaneously. In

order to express Lorentzian and Euclidean gravity in a unified way, we define the extrinsic curvature as[18]

$$\beta\hat{K}_a{}^i = \hat{A}_a{}^i - \hat{\Gamma}_a{}^i(\hat{\tilde{E}}), \tag{18}$$

where β can be taken as real and the double complex numbers. When taking $\beta = \pm i$, Eq.(18) corresponds to Lorentzian gravity, whereas $\beta = \pm 1$ or $\pm\epsilon$ correspond to Euclidean gravity.

For the discussion of the doubly conformally reality conditions, the key steps of the program may be stated as follows.

1. The reality conditions for $(\hat{\tilde{E}}_P)^a{}_i$ and $\hat{K}_a{}^i$

Since the densitized triad $(\hat{\tilde{E}}_P)^a{}_i$ is real in all the theories that we are studying, we require that $(\hat{\tilde{E}}_P)^a{}_i$ be Hermitian as part of the reality conditions. Thus

$$\begin{aligned}(\hat{\tilde{E}}^a{}_i)^* =& \overline{\Omega}^2(J)((\hat{\tilde{E}}_P)^a{}_i)^* \\ =& \Omega^{-2}(J)((\hat{\tilde{E}}_P)^a{}_i) = \Omega^{-4}(J)\hat{\tilde{E}}^a{}_i.\end{aligned} \tag{19}$$

Then, applying the correspondence principle, we take the transformation for Eqs.(1), and thus obtain the Lie algebra of the $\hat{A}_a{}^i$ and $\hat{\tilde{E}}^a{}_i$:

$$([\hat{A}_a{}^i(x), \hat{\tilde{E}}^b{}_j(y)]) = -\hbar\delta^b_a\delta^i_j\delta^3(x,y). \tag{20}$$

Taking the *-conjugate of Eq.(20), we arrive at the result

$$\begin{aligned}&([\beta\hat{K}_a{}^i(x), \hat{\tilde{E}}^b{}_j(y)])^* \\ =& -\hbar\delta^b_a\delta^i_j\delta^3(x,y) \\ =& -\Omega^{-4}(J)[\overline{\beta}(\hat{K}_a{}^i)^*(x), \hat{\tilde{E}}^b{}_j(y)].\end{aligned} \tag{21}$$

Then the reality conditions for $\hat{K}_a{}^i$ must be of the form

$$(\hat{K}_a{}^i)^* = -\beta^2\Omega^4(J)\hat{K}_a{}^i + f_a{}^i(\hat{\tilde{E}}), \tag{22}$$

where $f_a{}^i$ is a function of densitized triad (and its spatial derivatives) which is to be determined.

2. The reality conditions for the constraints

Using the same method as Marugan's,[17] we can derive the reality conditions for the constraints. The only difference is the value of the double conformal factor $\Omega(J)$.

After the complicated calculations, we can obtain the reality conditions for the constraints, i.e.,

$$\epsilon_{ij}{}^k f_a{}^j(\hat{\tilde{E}})\hat{\tilde{E}}^a{}_k = 0, \tag{23}$$

$$D_b(f_a{}^i(\hat{\tilde{E}})\hat{\tilde{E}}^b{}_i - f_c{}^i(\hat{\tilde{E}})\hat{\tilde{E}}^c{}_i\delta^b_a) = 0, \tag{24}$$

$$\begin{aligned}&(1-\Omega^8(J))q^2(\hat{\tilde{E}})R(\hat{\tilde{E}}) + 2f_a{}^i(\hat{\tilde{E}})f_b{}^j(\hat{\tilde{E}})\hat{\tilde{E}}^a{}_{[i}\hat{\tilde{E}}^b{}_{j]} \\ =& -4\Omega^4(J)f_a{}^i(\hat{\tilde{E}})\hat{K}_b{}^j\hat{\tilde{E}}^a{}_{[i}\hat{\tilde{E}}^b{}_{j]},\end{aligned} \tag{25}$$

where the brackets [ ] denote antisymmetrization, $R(\hat{\tilde{E}})$ is the operator associated with the scalar curvature operator of the 3-metric q_{ab}, and $q^2(\hat{\tilde{E}})$ corresponds to the determinant of the metric

$$q^2(\hat{\tilde{E}}) = \frac{1}{6}\epsilon^{ijk}\epsilon_{abc}\hat{\tilde{E}}^a{}_i\hat{\tilde{E}}^b{}_j\hat{\tilde{E}}^c{}_k. \tag{26}$$

Here, we have used the double conformal transformation for the classical quantity corresponding to Eq.(19).

$$\begin{aligned}\overline{\tilde{E}}^a_i \equiv& \Omega^{-4}(J)\tilde{E}^a_i, \\ R(\Omega^{-4}(J)\tilde{E}) \equiv& R(\Omega^{-4}(J)q_{ab}) \\ =\Omega^4(J)R(q_{ab}) \equiv& \Omega^4(J)R(\tilde{E}).\end{aligned} \tag{27}$$

In particular, Eq.(25) is functionally independent of the scalar and vector constraints.

3. Discussion of the double conformal factor

As pointed out in Ref.[17], the only way in which relation (25) can be satisfied is if both sides of the considered equation vanish, i.e. $f_a{}^i = 0$. Obviously, for $f_a{}^i = 0$, the conditions (22), (23) and (24) are immediately fulfilled. Thus, Eq.(25) can be written as

$$(1-\Omega^8(J))q^2(\hat{\tilde{E}})R(\hat{\tilde{E}}) = 0.$$

As is known, in the general case, $q^2(\hat{\tilde{E}})R(\hat{\tilde{E}}) \neq 0$. Thus, one has to demand that $\Omega^8(J) = 1$. In other words, once one assumes that the densitized triad $(\hat{\tilde{E}}_P)^a{}_i$ (that is real in the classical theory) is Hermitian, the compatibility of the reality conditions with the algebraic structures and constraints of the Ashtekar formulation restricts the double conformal factor to be $\Omega^8(J) = 1$. So it is easy to arrive at the physically meaningful solution, i.e., $\Omega^2 = \pm 1$, $\Omega^2 = \pm i$ and $\Omega^2 = \pm\epsilon$. Now let us discuss the set of reality conditions for the gravitational theory corresponding to different values of β and Ω.

4. Brief analysis of the physical meaning corresponding to various values of β and Ω

Provided that $\beta = i$, i.e., we discuss the doubly conformally reality conditions from the Lorentzian gravity.

(1) When $\Omega^2 = \pm 1$, because of $(\hat{K}_a{}^i)^* = \hat{K}_a{}^i$, $\hat{A}_a{}^i = \hat{\Gamma}_a{}^i(\hat{\tilde{E}}) + i\hat{K}_a{}^i$, we can arrive at the conclusion

$$(\hat{\tilde{E}}^a{}_i)^* = \hat{\tilde{E}}^a{}_i,\ (\hat{A}_a{}^i)^* = -\hat{A}_a{}^i + 2\hat{\Gamma}_a{}^i(\hat{\tilde{E}}). \tag{28}$$

These are is the well-known reality conditions[3,17] for Lorentzian gravity in the Ashtekar formulation. It can be shown that these conditions are the straightforward translation to the algebra of operators for the requirements that the classical densitized triad must be real (so that the metric $\tilde{\tilde{q}}^{ab} = \tilde{E}^a{}_i\tilde{E}^{ai}$ is positive) and the real part of the the classical Ashtekar connection given by the SO(3) connection has to be real.

(2) When $\Omega^2 = \pm i$, because of $(\hat{K}_a{}^i)^* = -\hat{K}_a{}^i$, $\hat{A}_a{}^i = \hat{\Gamma}_a{}^i(\hat{\tilde{E}}) + i\hat{K}_a{}^i$, this corresponds to the reality conditions for Euclidean gravity in the Ashtekar formulation

$$(\hat{\tilde{E}}^a{}_i)^* = -\hat{\tilde{E}}^a{}_i,\ (\hat{A}_a{}^i)^* = \hat{A}_a{}^i. \qquad (29)$$

In fact, it is not difficult to show that reality conditions (29) are simply the direct translation of the complex conjugation relations for real Euclidean gravity in the Ashtekar formulation.

(3) When $\Omega^2 = \pm\epsilon$, we can obtain the same conclusion in form as that of case (1). Note that although the result is the same as Eq.(28) in form, the values of the triad are taken in the hyperbolic complex number ring.[16]

Obviously, by use of the double conformal factor $\Omega(J)$, we can express the set of reality conditions for Lorentzian and Euclidean gravity in a unified way as follows

$$\begin{aligned}(\hat{\tilde{E}}^a{}_i)^* =& \Omega^4\hat{\tilde{E}}^a{}_i,\ (\hat{A}_a{}^i)^* \\ =& -\Omega^4\hat{A}_a{}^i + (1+\Omega^4)\hat{\Gamma}_a{}^i(\hat{\tilde{E}}). \qquad (30)\end{aligned}$$

It follows that we have extended the results of the previous theory.[1,17]

Similarly, we can also discuss the reality conditions starting from Euclidean gravity, i.e. taking $\beta = 1$ and ϵ under the double conformal transformation by the same method as the above. Finally, it is not difficult to show that after the transformation the set of reality conditions for Lorentzian and Euclidean gravity (corresponding to different values of β and Ω) can be expressed in a unified way by Eq.(30). It is just the results given by us using the generalized Marugan's method.

5. Conclusions

Our main work presented in this paper is the introduction of the double conformal transformation that has the merit of deriving the set of reality conditions for Lorentzian and Euclidean gravity simultaneously. Under this transformation, if one requires that the classical real triad is represented by a Hermitian operator, the compatibility of the *-operation with the algebra of commutators and constraints restricts the admissible double conformal factors to be either real or double complex unit. When the double conformal factors are taken as the real and hyperbolic complex unit, one obtains the well-known set of reality conditions for Lorentzian gravity; when the double conformal factor are taken as the elliptic complex unit, one can obtain the set of reality conditions for Euclidean gravity. Furthermore, the set of reality conditions for Lorentzian and Euclidean gravity can be expressed in a unified way by use of the DCFM.

References

[1] Ashtekar A 1986 *Phys. Rev. Lett.* **57** 2244
[2] Ashtekar A 1987 *Phys. Rev.* D **36** 1587
[3] Ashtekar A 1991 *Lectures on Non-Perturbative Canonical Gravity* eds L.Z.Fang and R.Ruffini (Singapore: World Scientific)
[4] Rovelli C and Smolin L 1988 *Phys. Rev. Lett.* **61** 1155
[5] Rovelli C and Smolin L 1990 *Nucl. Phys.* B **331** 80
[6] Barbero J F 1994 *Phys. Rev.* D **49** 6935
[7] Barbero J F 1995 *Phys. Rev.* D **51** 5498
[8] Barbero J F 1995 *Phys. Rev.* D **51** 5507
[9] Ashtekar A 1996 *Phys. Rev.* D **53** 2865
[10] Holst S 1996 *Phys. Rev.* D **53** 5966
[11] Barbero J F 1996 *Phys. Rev.* D **54** 1492
[12] Ashtekar A, Rovelli C and Smolin L 1992 *Phys. Rev. Lett.* **63** 237
[13] Li X Z, Yuan N Y , Liu D J and Hao J G 2000 *Acta Phys. Sin.* **49** 1031 (in Chinese)
[14] Zhao Z, Pei S Y and Liu L 1999 *Acta Phys. Sin.* **48** 2004 (in Chinese)
[15] Wu Y B 1998 *Int. J. Theor. Phys.* **37** 2127
[16] Zhong Z Z 1985 *J. Math. Phys.* **26** 2589
[17] Mean Marugan G A 1993 *Preprint* gr-qc/9311020
[18] Wu Y B and Li J L 2001 *Chin. Phys. Lett.* **18** 328
[19] Li J L and Wu Y B 2001 *Acta Phys. Sin.* **50** 411 (in Chinese)
[20] Wu Y B and Gui Y X 2001 *Chin. Phys.* **10** 902