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The double complex Ashtekar gravitational theory and the corresponding constraint equations with the cosmological term*

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We establish the double complex Ashtekar gravitational theory with the cosmological term. In particular, by performing the 3+1 decomposition of the double Ashtekar action containing the cosmological term to pass on the Hamiltonian framework, the double Ashtekar constraint equations are derived, which respectively correspond to Lorentzian and Euclidean gravity.

Keywords: Ashtekar gravity, constraint equations, reality conditions
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1. Introduction

It is well known that the Ashtekar gravitational theory [1] makes essential use of the chiral decomposition of the connection one-forms of the local Lorentz group. In this theory, at least for Einstein manifolds without matter, the full set of field equations of general relativity can not only be recovered, but can also be simplified by use of only one of the two chiral projections of these connection forms.[1−5] We may either use the self-dual or the anti-self-dual connections and their respective conjugate variables. However, these (anti-)self-dual variables must be complex for the Lorentzian signature of the metric according to the definition of the dual transformations in differential geometry. Therefore, the Ashtekar gravitational theory is a complex theory. Hence, reality conditions must be imposed on the complex Ashtekar variables in order to recover the real Einstein gravitational theory. Since the reality conditions can bring some new difficulties to the quantization for gravity, this has become an important research subject in Ashtekar gravitational theory. However, we know that some authors[6−8] have discussed Euclidean gravitational theory (with Euclidean signature of the metric), in which the (anti-)self-dual variables must be real variables. But mathematically they may be also hyperbolic complex variables.[9−11] As has been discussed by Wu and Gui,[12] by using the double complex function method[13] we have given some definitions of double dual transformations and double (anti-)self-dual variables, and hence we have constructed a double complex (anti-)self-dual gravitational action with the cosmological term. However, in this paper we discuss whether we can put this action in Hamiltonian form and obtain corresponding constraint equations in 3+1 gravity.

For the convenience of the following discussion, here we review some necessary results and symbols.[12,13] We let $M(\tilde{J})$ be a double real 4-manifold $M(\tilde{J}) = (M_C, M_H)$ instead of the usual real 4-manifold (here, $\tilde{J}$ is the double pure imaginary unit, i.e., $\tilde{J} = i$ or $\tilde{J} = \epsilon$). For any point p in $M(\tilde{J})$ there is a double complex internal space, which includes an internal Lorentzian space for M_C with Lorentzian signature $\eta_L = \mathrm{diag}(-1, +1, +1, +1)$, transformed by the local Lorentz group $SO(1,3)$, and an internal Euclidean space for M_H with Euclidean signature $\eta_E = \mathrm{diag}(+1, +1, +1, +1)$, transformed by the local Euclidean group $SO(4)$. Here we denote in a unified way $SO(1,3)$ and $SO(4)$ as $SO(4, \eta(\tilde{J}))$, where $\eta(\tilde{J}) = (\eta(\tilde{J} = i), \eta(\tilde{J} = \epsilon)) = (\eta_C, \eta_H) = (\eta_L, \eta_E)$. Thus, when taking $\tilde{J} = i$, $SO(4, \eta(\tilde{J})) \equiv SO(1,3)$, and when $\tilde{J} = \epsilon$, $SO(4, \eta(\tilde{J})) \equiv SO(4)$.

Penrose's abstract index notation will be used throughout. Space-time and spatial tensor indices are both denoted by $a, b, c, \ldots$, running from 0 to 3 and 1 to

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3, respectively; and internal indices and their spatial indices will denoted by $I, J, K, ...$ and $i, j, k, ...$, running from 0 to 3 and 1 to 3, respectively. Thus the double Ashtekar action can be written as[12]

$$S^{\pm}(\tilde{J}) = \frac{1}{16\pi G}\int_{M(\bar{J})} \Sigma^{\pm IJ}(\tilde{J}) \wedge F^{\pm}_{IJ}(\tilde{J}) \pm \tilde{J}\frac{\lambda}{3(8\pi G)}\int_{M(\bar{J})} \Sigma^{\pm IJ}(\tilde{J}) \wedge \Sigma^{\pm}_{IJ}(\tilde{J}), \quad (1)$$

where λ is the cosmological constant, while the (anti-)self-dual operations of the double curvature two-form $F_{IJ}(\tilde{J})$ of $\omega_{IJ}(\tilde{J})$ and the double two-form $\Sigma_{IJ}(\tilde{J}) = e_I \wedge e_J(\tilde{J})$ (here $e_I(\tilde{J})(I = 0, ..., 3)$ denote tetrad one-forms) are defined as

$$\begin{aligned} F^{\pm}_{IJ}(\tilde{J}) =& \frac{1}{2}(F_{IJ}(\tilde{J}) \pm \tilde{J}\tilde{J}^2 F_{IJ} * (\tilde{J})), \\ \Sigma^{\pm}_{IJ}(\tilde{J}) =& \frac{1}{2}\Big(e_I(\tilde{J}) \wedge e_J \\ & \pm \tilde{J}\tilde{J}^2 \frac{1}{2}\epsilon_{IJ}{}^{KL}(\tilde{J}) e_K(\tilde{J}) \wedge e_L(\tilde{J})\Big), \end{aligned} \quad (2)$$

where $F_{IJ}(\tilde{J}) = d\omega_{IJ}(\tilde{J}) + \omega_{IK}(\tilde{J}) \wedge \omega^K_J(\tilde{J})$.

2. The constraint equations with the cosmological term

In order to perform the 3+1 decomposition of the double Ashtekar action $S^{\pm}(\tilde{J})$, we assume that $M(\tilde{J}) = \sum(\tilde{J}) \times R(\tilde{J})$ for some double space-like submanifold $\Sigma(\tilde{J})$. Since we have the double space-time metric $g_{ab}(\tilde{J})$ on $M(\tilde{J})$, we can introduce a double unit covariant normal $n^a(\tilde{J})$ and its associated future-pointing time-like vector field $n^a(\tilde{J}) = g^{ab}(\tilde{J})\ n_b(\tilde{J})$. Note that $n^a(\tilde{J})n_a(\tilde{J}) = \tilde{J}^2$, in which $\tilde{J} = i$ and $\tilde{J} = \epsilon$ respectively correspond to the time-like curves of the Lorentzian and Euclidean signature of metric. In addition, we define the double induced metric $q_{ab}(\tilde{J})$, the lapse $N(\tilde{J})$, and shift $N^a(\tilde{J})$ via

$$\begin{aligned} q_{ab}(\tilde{J}) :=& q^c_a(\tilde{J}) q^d_b(\tilde{J}) g_{cd}(\tilde{J}), \\ N(\tilde{J}) :=& \tilde{J}^2 t^a n_a(\tilde{J}), \\ N^a(\tilde{J}) :=& t^b q^a_b(\tilde{J}). \end{aligned} \quad (3)$$

Thus, the double space-time metric $g_{ab}(\tilde{J})$ on $M(\tilde{J})$ can be expressed as

$$g_{ab}(\tilde{J}) = \begin{pmatrix} N_c(\tilde{J})N^c(\tilde{J}) + \tilde{J}^2 N^2(\tilde{J}) & N_b(\tilde{J}) \\ N_a(\tilde{J}) & q_{ab}(\tilde{J}) \end{pmatrix}, \quad (4)$$

$$g^{ab}(\tilde{J}) = \begin{pmatrix} \dfrac{\tilde{J}^2}{N^2(\tilde{J})} & -\dfrac{\tilde{J}^2 N^b(\tilde{J})}{N^2(\tilde{J})} \\ -\dfrac{\tilde{J}^2 N^a(\tilde{J})}{N^2(\tilde{J})} & q^{ab}(\tilde{J}) + \tilde{J}^2 \dfrac{N^a(\tilde{J})N^b(\tilde{J})}{N^2(\tilde{J})} \end{pmatrix}. \quad (5)$$

When taking $\tilde{J} = i, g_{ab}(\tilde{J} = i) = g_{ab}(C)$ in Eq.(4) is just compatible with the Arnowitt–Deser–Misner (ADM) [14] decomposition of the metric with Lorentzian signature. Furthermore, by means of $g_{ab}(\tilde{J}) = e_{Ia}(\tilde{J}) e^I_b(\tilde{J})$, we can equally express Eq.(4) with the components of the double tetrad and its inverse in the following form

$$e_{aI}(\tilde{J}) = \begin{pmatrix} -\tilde{J}^2 N(\tilde{J}) & 0 \\ -N^b(\tilde{J}) e_{ib}(\tilde{J}) & e_{ai}(\tilde{J}) \end{pmatrix},$$

$$e^a_I(\tilde{J}) = \begin{pmatrix} -\tilde{J}^2 N^{-1}(\tilde{J}) & 0 \\ \tilde{J}^2 \dfrac{N^a(\tilde{J})}{N(\tilde{J})} & e^a_i(\tilde{J}) \end{pmatrix}, \quad (6)$$

where $e_{aI}(\tilde{J})$ and $e^b_I(\tilde{J})$ satisfy $e_{Ia}(\tilde{J})\ e^{Ib}(\tilde{J}) = \delta^b_a(\tilde{J})$ and $e^a_I(\tilde{J}) = \eta_{IJ}(\tilde{J}) e^{Ja}(\tilde{J})$. With this decomposition, the double Ashtekar action $S^{\pm}(\tilde{J})$ in Eq.(1) can be rewritten by a straightforward calculation

$$\begin{aligned} S^{\pm}(\tilde{J}) =& \frac{1}{8\pi G}\int_{M(\bar{J})} \mathrm{d}^4x \Big\{ \pm 2\tilde{J}\tilde{J}^2 \big[N^a(\tilde{J})\tilde{\sigma}^{bi}(\tilde{J}) F^{\pm}_{abi}(\tilde{J}) \\ & + \tilde{\sigma}^{ai}(\tilde{J})\dot{\omega}^{\pm}_{ia}(\tilde{J}) + \omega^{\pm}_{i0}(\tilde{J}) D_a \tilde{\sigma}^{ai}(\tilde{J})\big]\Big\} \\ & - \frac{1}{16\pi G}\int_{M(\bar{J})} \mathrm{d}^4x \Big\{ \overset{N}{\sim}\big[\epsilon_{ijk}(\tilde{J})\tilde{\sigma}^{ai}(\tilde{J})\tilde{\sigma}^{bj}(\tilde{J}) F^{\pm k}_{ab}(\tilde{J}) \\ & + \frac{\lambda}{3}\epsilon_{ijk}(\tilde{J}) \overset{\epsilon}{\sim}_{abc}(\tilde{J})\tilde{\sigma}^{ai}(\tilde{J})\tilde{\sigma}^{bj}(\tilde{J})\tilde{\sigma}^{ck}(\tilde{J})\big]\Big\} + \text{boundary terms}, \end{aligned} \quad (7)$$

where

$$\begin{aligned} \overset{N}{\sim}(\tilde{J}) \equiv& \det\left(e_{ai}(\tilde{J})\right)^{-1} N(\tilde{J}), \\ \tilde{\sigma}^{ia}(\tilde{J}) \equiv& \frac{1}{2}\epsilon^{ijk}(\tilde{J})\tilde{\epsilon}^{abc}(\tilde{J}) e_{bj}(\tilde{J}) e_{ck}(\tilde{J}). \end{aligned} \quad (8)$$

The tildes above and below the variables indicate that they are tensor densities of weight 1 and -1, respectively. From Eq.(7) we can see that the double Ashtekar action $S^{\pm}(\tilde{J})$ can be decomposed in 3+1

form, and the double conjugate variables to the dynamical variables $\omega_{ai}^{\pm}(\tilde{J})$ are just $\frac{\pm 2\tilde{J}\tilde{\sigma}^{ia}(\tilde{J})}{8\pi G}$, which satisfy the relations

$$\{\tilde{\sigma}^{ia}(\overrightarrow{x},t;\tilde{J}),\omega_{jb}^{\pm}(\overrightarrow{y},t;\tilde{J})\} = \pm 8\pi G\tilde{J}\delta_j^i\delta_b^a\delta^3(\overrightarrow{x}-\overrightarrow{y}); \quad (9)$$

while the double variables $\omega_{i0}^{\pm}(\tilde{J})$, $N^a(\tilde{J})$ and $\tilde{N}(\tilde{J})$ are clearly Lagrange multipliers. Thus we can obtain the following constraint equations

$$2\tilde{J}D_a\tilde{\sigma}_i^a(\tilde{J}) \approx 0, \quad (10)$$

$$2\tilde{J}\tilde{\sigma}_i^b(\tilde{J})F_{ab}^{i\pm}(\tilde{J}) \approx 0, \quad (11)$$

$$\epsilon^{ijk}(\tilde{J})\tilde{\sigma}_i^a(\tilde{J})\tilde{\sigma}_j^b(\tilde{J})\left[F_{abk}^{\pm}(\tilde{J}) + \frac{\lambda}{3}\overset{\epsilon}{\sim}_{abc}(\tilde{J})\tilde{\sigma}_k^c(\tilde{J})\right] \approx 0, \quad (12)$$

which are just the double Ashtekar constraint equations with the cosmological term. When taking $\tilde{J}=i$, Eqs.(10)–(12) are just the results given by Ashtekar.[1] When taking $\tilde{J}=\epsilon$, they are the new constraints given by us, which are here called the hyperbolic Ashtekar constraints, and take values on a hyperbolic complex number ring.[13] If the cosmological constant λ is taken to be zero, Eqs.(10)–(12) change into

$$\tilde{G}_i \equiv 2\tilde{J}D_a\tilde{\sigma}_i^a(\tilde{J}) \approx 0, \quad (13)$$

$$\tilde{V}_a \equiv 2\tilde{J}\tilde{\sigma}_i^b(\tilde{J})F_{ab}^{i\pm}(\tilde{J}) \approx 0, \quad (14)$$

$$\tilde{\tilde{S}} \equiv \epsilon^{ijk}(\tilde{J})\tilde{\sigma}_i^a(\tilde{J})\tilde{\sigma}_j^b(\tilde{J})F_{abk}^{\pm}(\tilde{J}) \approx 0. \quad (15)$$

These are just the well-known constraints.[4,14] In this case, the Hamiltonian can be expressed as

$$H = \int_\Sigma \mathrm{d}^3x\left(\frac{1}{2}\overset{N}{\sim}\tilde{\tilde{S}} - \mathrm{i}N^a(\tilde{V}_a - A_a^i\tilde{G}_i) + \mathrm{i}N^i\tilde{G}_i\right). \quad (16)$$

3. Conclusions and discussion

So far, we have established the double complex Ashtekar gravitational theory with the cosmological term. In particular, by performing the 3+1 decomposition of the double Ashtekar action $S^{\pm}(\tilde{J})$ with the cosmological term to pass on the Hamiltonian framework, we have obtained the double Ashtekar constraint equations (10)–(12), including the ordinary and hyperbolic Ashtekar constraint equations which correspond respectively to Lorentzian and Euclidean gravity. In the special case of taking the cosmological constant $\lambda=0$, these constraints can turn back to the previous well-known ones.

In addition, we know that self-dual connection in the Ashtekar's theory are complex variables, hence Ashtekar's theory is a complex theory. However, as a gravitational theory, it must degenerate into the Einstein's general relativity in some special conditions. So one imposes the reality conditions on the Ashtekar complex variables in order to recover the usual real formulation of general relativity for space-times with Lorentzian signatures. But Ashtekar gave the denotation of the reality conditions just in form,[15,16] i.e. $(\tilde{\sigma}^a{}_i\tilde{\sigma}^{bi})$ is real and positive, and its derivative with respect to time is also real. In fact, the reality conditions given by Ashtekar are not valid in the canonical quantization program.[16] Therefore, at present the problem has become an important research subject of the Ashtekar's theory. As for the study of reality conditions, there have been many methods and results.[16–19] For example, Marugan[16] successfully derived the reality conditions in the Ashtekar formulation by use of the conformal transformation. But he only started from Lorentzian gravity to discuss the reality conditions, and we find that there is a limitation for the conformal factor Ω in his method.

We believe that we can generalize Marugan's method by introducing a double conformal transformation

$$\mathrm{d}s^2 = \Omega^2(\tilde{J})(\mathrm{d}s_P)^2. \quad (17)$$

Here, $\Omega(\tilde{J})$ is a double complex function whose absolute value is equal to unity, i.e. $\Omega = e^{J\theta}, \theta \in [0,2\pi)$, $\mathrm{d}s_P$ is the space-time line element on a space-times manifold M_p with Lorentzian or Euclidean signature. At present, by means of the double complex function method we have given the relations between constraints such as Eqs.(13)–(15) under the double conformal transformation (17) as follows

$$\begin{aligned}\tilde{G}_i &\equiv \Omega^2(\tilde{J})(\tilde{G}_P)_i = 0,\\ \tilde{V}_a &\equiv \Omega^2(\tilde{J})(\tilde{V}_P)_a = 0,\\ \tilde{\tilde{S}} &\equiv \Omega^4(\tilde{J})\tilde{\tilde{S}}_P = 0,\end{aligned} \quad (18)$$

which can lay a good foundation for deriving the reality conditions in the following study. At present we are undertaking work to find how we study the problem of the reality conditions as well as what the concrete forms about them are. We will give these results soon and publish them in the forthcoming paper.

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