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An extension of the direct method and similarity reductions of a generalized Burgers equation with an arbitrary derivative function*

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In this paper, we extend the well-known direct method proposed by Clarkson and Kruskal for finding similarity reductions of partial differential equations. It follows that some new similarity reductions of the generalized Burgers equation, such as travelling wave reduction, logarithmic reduction, power reduction, rational fractional reduction, etc, are derived, in which some of these cannot be obtained solely by using the direct method. The similarity reductions obtained are interpreted by the nonclassical symmetry Lie group.

Keywords: similarity reduction, Burgers equation, nonclassical symmetry Lie group
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1. Introduction

A powerful method for finding similarity reductions of a given partial differential equation (PDE) is to use the Lie group method of infinitesimal transformations originally developed by Lie.[1] Though the method is entirely algorithmic, it involves a large amount of tedious algebra and auxiliary calculations, which are virtually unmanageable manually. Bluman and Cole[2] proposed a generalization of the Lie method, which they called the nonclassical method of group-invariant solutions, and this has been generalized by Olver and Rosenau. All these methods determine Lie point transformations of a given partial differential equation, that is, transformations depending only on the independent and dependent variables. Noether recognized that the Lie method could be generalized by allowing the transformations to depend on the derivatives of the dependent variable as well as the independent and dependent variables[3]. The associated symmetries, called Lie–Backlund symmetries, can also be determined by an algorithmic method.

Bluman and Cole[2] have introduced an algorithmic method which yields new classes of symmetries of a given partial differential equation that are neither Lie point nor Lie–Backlund symmetries. A common characteristic of all these methods for finding symmetries and associated similarity reductions of a given partial differential equation is the use of group theory.

Clarkson and Hood[4] and Clarkson and Kruskal[5] have developed a method of deriving similarity reductions of partial differential equations and they obtained the similarity reductions of the Boussinesq equation; we call this approach the direct method. This method does not use group theory and can obtain new similarity solutions which cannot be obtained by using a standard group method. The basic idea of the direct method is to seek a reduction of a given partial differential equation in the form

$$u(x,t) = U(x,t,w(z(x,t))), \tag{1}$$

which is the most general form for a similarity reduction. Substituting Eq.(1) into the partial differential equation and demanding that the result be an ordinary differential equation (ODE) for $w(z)$, we impose conditions upon U and its derivatives that enable us to solve for U. For most partial differential equations, it is sufficient to take Eq.(1) in the special form[5]

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)). \tag{2}$$

Using this direct method, some authors have obtained many similarity reductions for various partial differential equations.[6–8]

In this paper, we develop the form of Eq.(2) into the following

$$h(u) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{3}$$

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where $h(u)$ is a derivative function with respect to u. It is easy to find that Eq.(3) is an extended form of Eq.(2). Furthermore, we use Eq.(3) to obtain some new similarity reductions of a generalized Burgers equation as

$$u_t + u_x + u_{xx} + h(u)u_x = 0, \tag{4}$$

which cannot be obtained just by the direct method proposed by Clarkson and Kruskal.

Finally, we present the similarity reductions obtained by the nonclassical Lie symmetry group interpretation.

2.Some new similarity reductions of Eq.(4)

In what follows, we consider two distinct cases.

2.1. For the case $z_x \neq 0$.

Substituting Eq.(3) into Eq.(4) yields

$$\begin{aligned}&\beta z_x^2 w_{xx} + (\beta z_x + 2\beta_x z_x + \beta z_{xx} + \alpha\beta z_x + \beta z_t)w_z \\ &+ \beta^2 z_x w w_x + \beta\beta_x w^2 + (\alpha\beta_x + \beta\alpha_x + \beta_x \\ &+ \beta_{xx} + \beta_t)w + \alpha_t + \alpha_x\alpha_{xx} + \alpha\alpha_x \\ =&\frac{h''}{h'^2}(\alpha_x^2 + \beta_x w^2 + \beta^2 z_x^2 w_z^2 + 2\alpha_x\beta_x w \\ &+ 2\alpha_x\beta z_x w_x + 2\beta\beta_x w w_x).\end{aligned} \tag{5}$$

In order for Eq.(5) to be an ordinary differential equation for $w(z)$, the ratios of coefficients of different derivatives and powers of $w(z)$ have to be functions of z only. To this end, there are three selections:

(i) if $\alpha(x,t)$ has the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Omega(z)$ then we can take $\Omega \equiv 0$ (replacing $w(z)$ by $w(z) - \Omega(z)$);

(ii) if $\beta(x,t)$ has the form $\beta = \beta_0(x,t)\Omega(z)$ then we can take $\Omega \equiv 1$ (replacing $w(z)$ by $w(z)/\Omega(z)$);

(iii) if $z(x,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,t)$ where $\Omega(z)$ is an invertible function, then we can take $\Omega(z) = z$ (let $z \to \Omega^{-1}(z)$).

We now proceed to determine some similarity reductions of Eq.(4).

In the following, there are a few subclasses to consider.

1) For an arbitrary $h(u)$, if $\dfrac{h''}{h'^2} = f(w)$, we can take $u = w$, then $w(z)$ satisfies

$$w_{zz} + ww_z - f(w)w_z^2 = 0, \tag{6}$$

which is equivalent to a travelling wave reduction.

2) If at least one of α and β is not a constant, we let $\left(\dfrac{h''}{h'^2}\right)^p = \dfrac{\sum\limits_{j=0}^{m} a_j h^j}{\sum\limits_{i=0}^{n} b_i h^i}$. Then Eq.(5) reads

$$\begin{aligned}&\left[\sum_{i=0}^{n} b_i(\alpha+\beta w)^i\right][\beta z_x^2 w_{zz} + \beta^2 z_x w w_z + (\alpha\beta z_x + \beta z_{xx} + 2\beta_x z_x + \beta z_x + \beta z_t)w_z + \beta\beta_x w^2 \\ &+ (\beta_{xx} + \alpha_x\beta + \alpha\beta_x + \beta_x + \beta_t)w + \alpha_x + \alpha_{xx} + \alpha\alpha_x + \alpha_t]^p \\ =&\left[\sum_{i=0}^{m} \alpha_i(\alpha+\beta w)^i\right](\alpha_x^2 + 2\alpha_x\beta_x w + 2\alpha_x\beta z_x w_z + 2\beta\beta_x z_x w w_z + \beta_x^2 w^2 + \beta^2 z_x^2 w_z^2)^p.\end{aligned} \tag{7}$$

(a) If α_j=0, j=0,1,2, Λ, m, then we can let $h = \lambda u$, where λ is a constant, Eq.(4) reduces to

$$u_t + u_x + u_{xx} + \lambda u u_x = 0, \tag{8}$$

which is called a generalized Burgers equation. Its similarity reductions and some exact solutions have been presented in Ref.[9].

(b) If $m = n = 0$, $a_0 \neq 0$, $p = b_0 = 1$, then $\dfrac{h''}{h'^2} = a_0$, leading to

$$h = -\frac{1}{a_0}\ln(u+k),$$

and Eq.(7) reduces to

$$\begin{aligned}&\beta z_x^2 w_{zz} + \beta^2 z_x w w_z + (\alpha\beta z_x + \beta z_{xx} + 2\beta_x z_x + \beta z_x + \beta z_t)w_z + \beta\beta_x w^2 \\ &+ (\beta_{xx} + \alpha_x\beta + \alpha\beta_x + \beta_x + \beta_t)w + \alpha_x + \alpha_{xx} + \alpha\alpha_x + \alpha_t \\ =&a_0(\alpha_x^2 + 2\alpha_x\beta_x w + 2\alpha_x\beta z_x w_z + 2\beta\beta_x z_x w w_z + \beta_x^2 w^2 + \beta^2 z_x^2 w_z^2).\end{aligned} \tag{9}$$

We use the coefficient of w_{zz} as the normalizing coefficient, and set

$$\begin{aligned}\beta^2 z_x =&\beta^2 z_x^2 \Gamma_1(z), \beta\beta_x - a_0\beta_x^2 = \beta^2 z_x^2 \Gamma_2(z),\\ &\alpha\beta z_x + \beta z_{xx} + 2\beta_x z_x + \beta z_x + \beta z_t - 2a_0\alpha_x\beta z_x = \beta^2 z_x^2 \Gamma_3(z),\\ &\beta_{xx} + \alpha_x\beta + \alpha\beta_x + \beta_x - 2\alpha_x\beta_x + \beta_t = \beta^2 z_x^2 \Gamma_4(z),\\ &\beta^2 z_x - 2a_0\beta\beta_x z_x = \beta^2 z_x^2 \Gamma_5(z), \beta z_x^2 = \beta^2 z_x^2 \Gamma_6(z),\\ &\alpha_x + \alpha_{xx} + \alpha\alpha_x - a_0\alpha_x^2 + \alpha_t = \beta^2 z_x^2 \Gamma_6(z). \end{aligned} \tag{10}$$

Solving Eq.(10) yields

$$\Gamma_1(z) = \Gamma_5(z) = \Gamma_7(z) = 1, \Gamma_2(z) = \Gamma_3(z) = \Gamma_4(z) = 0, \Gamma_6(z) = c_0.$$

The similarity reduction of Eq.(4) reads

$$-\frac{1}{a_0}\ln(u+k) = c_0 t + w(z), z = x + \frac{c_0}{2}t^2 + c_3 t + c_4,$$

where $w(z)$ satisfies the ODE: $w_{zz} + ww_z - a_0 w_z^2 + c_0 = 0$.

(c) Taking $m = 0, n = p = 1$, $\dfrac{h''}{h'^2} = \dfrac{a}{h+q}$, we let $\alpha + q = 0, q = 0$, where $a = \dfrac{a_0}{b_1}, q = \dfrac{b_0}{b_1}$. When $a \neq 1$, solving $\dfrac{h''}{h'^2} = \dfrac{a}{h}$ yields $h = c_5 u^{\frac{1}{1-a}}$, then Eq.(7) reduces to

$$\begin{aligned}&\beta z_x^2 w_{zz} + \beta^2 z_x w w_z + (\beta z_{xx} + 2\beta_x z_x + \beta z_x + \beta z_t)w_z + \beta\beta_x w^2 + (\beta_{zz} + \beta_x + \beta_t)w\\ =&\frac{a}{\beta w(z)}(2\beta\beta_x z_x w w_z + \beta_x^2 w^2 + \beta^2 z_x^2 w_z^2).\end{aligned}$$

Similar to Eq.(9), we have

$$\begin{aligned}\beta^2 z_x^2 =&\beta^2 z_x^2 \Gamma_1(z), \beta^3 z_x = \beta^2 z_x^2 \Gamma_2(z), \beta^2 z_{xx} + 2\beta\beta_x z_x + \beta^2 z_t - 2\alpha\beta\beta_x z_x\\ =&\beta^2 z_x^2 \Gamma_3(z), \beta^2\beta_x = \beta^2 z_x^2 \Gamma_4(z), \beta\beta_{xx} + \beta\beta_x + \beta\beta_t - \alpha\beta_x^2 = \beta^2 z_x^2 \Gamma_5(z).\end{aligned}$$

Solving the above equation yields $\Gamma_1(z) = \Gamma_2(z) = \Gamma_5(z) = 1, \Gamma_3(z) = z, \Gamma_4(z) = 0$. Thus we obtain the similarity reduction of Eq.(4) as follows

$$cu^{\frac{1}{1-a}} = \theta(t)w(z), z = \theta(t)x + \sigma(t), \theta' = \theta^3, \sigma' - \theta^2\sigma + \theta = 0,$$

where $w(z)$ satisfies $ww_{zz} + w^2 w_z + zww_z + w^2 - aw_z^2 = 0$. When $a = 1, h = c_7 \mathrm{e}^{c_6 u}$ the similarity reduction of Eq.(4) reads

$$c_7 \mathrm{e}^{c_6 u} = \theta(t)w(z), \quad z(x,t) = \theta(t)x + \sigma(t), \quad \theta' = \theta^3, \quad \sigma' - \theta^2\sigma + \theta = 0,$$

where $w(z)$ satisfies $ww_{zz} + w^2 w_z + zww_z + w^2 - aw_z^2 = 0$.

2.2. For the case $z_x = 0$.

Let the similarity reduction of Eq.(4) be in the form

$$h(u) = \alpha(x,t) + \beta(x,t)w(t). \tag{11}$$

Substitution of Eq.(11) into Eq.(4) yields

$$\begin{aligned}&\beta w_z + (\beta_t + \beta_x + \beta_{xx} + \beta\alpha_x + \alpha\beta_x)w + \alpha_t + \alpha_x + \alpha_{xx} + \alpha\alpha_x + \beta\beta_x w^2\\ =&\frac{h''}{h'^2}(\alpha_x^2 + 2\alpha_x\beta_x w + \beta_x^2 w^2).\end{aligned} \tag{12}$$

In order for Eq.(12) to be an ODE for $w(t)$, three types of similarity reductions of Eq.(4) are obtained:

1) $h(u)=\dfrac{2}{x-t+c}$, where $h(u)=\lambda u$, i.e. Eq.(4) reduces to Eq.(8), as discussed above;

2) $h(u)=\dfrac{2}{x-t}+w(t)$, where $h=-\dfrac{1}{a_0}\ln(u+k)$, and $w(t)$ satisfies $w'-\dfrac{2}{(x-t)^2}w-\dfrac{4a_0}{(x-t)^4}=0$;

3) $h(u)=xw(t)$, where $h(u)$ is given by

$$\int_a^{h(u)} \mathrm{e}^{-\frac{a_0}{2}s^2}\,\mathrm{d}s=\lambda u+k, a_0\neq 0, \tag{13}$$

and $w(t)$ satisfies

$$xw'+w+xw^2-a_0w^3=0.$$

3. Nonclassical symmetry group reductions

Bluman *et al.* once proposed the method of nonclassical symmetry group[10,11] for similarity reductions of ODEs and PDEs. Lou[12] developed the method, especially for the case of $z_x=0$, to obtain many similarity solutions to PDEs; this is called the Lou method. In the following, the similarity reductions obtained above are to be interpreted by the method for nonclassical symmetry Lie group and Lou's approach.

We consider the one-parameter (ε) Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{cases}\tilde{x}=x+\varepsilon X(x,t,u)+O(\varepsilon^2),\\ \tilde{t}=t+\varepsilon T(x,t,u)+O(\varepsilon^2),\\ \tilde{u}=u+\varepsilon U(x,t,u)+O(\varepsilon^2),\end{cases} \tag{14}$$

with the infinitesimal generator $\Pi=X(x,t,u)\dfrac{\partial}{\partial x}+T(x,t,u)\dfrac{\partial}{\partial t}+U(x,t,u)\dfrac{\partial}{\partial u}$. Now we consider two distinct cases.

3.1. $T\neq 0$. Without loss of generality, we take $T=1$ using the method proposed by Bluman and Cole and the infinitesimal criterion[10,11] for Eq.(4) which leads to the following determining equations

$$\begin{aligned}&X_{uu}=0,\\ &U_{uu}-2XX_u+2X_u-2X_{xu}+2hX_u=0,\\ &-X_t+2X_uU+2U_{xu}-X_{xx}\\ &+(1+h)X_x+h'U-2XX_x=0,\\ &U_t+U_{xx}+2UX_x+(1+h)U_x=0.\end{aligned} \tag{15}$$

It is not very easy to obtain all solutions of Eqs.(15). However, some solutions, in the following cases, can be obtained.

3.1a. Taking $X=c$=constant, $U=0, f$ arbitrary, it follows that the similarity reduction 1) in 2.1 is obtained.

3.1b. Taking $h''=0$, i.e. $h=\lambda u$ then Eq.(4) is reduced to Eq.(8), and its similarity reduction was given in Ref.[9], corresponding to the similarity reduction 2) (a) in 2.1.

3.1c. Taking $h=-\dfrac{1}{a_0}\ln(u+k), X=-\dfrac{t}{a_0}, U=u+k$ then it is equivalent to the similarity reduction 2) (b) in 2.1.

3.1d. Taking $h=c_5u^{\frac{1}{1-a}}, X=\dfrac{1}{2}\left(\dfrac{x}{t}+1\right), U=\dfrac{a-1}{2t}$ then it is equivalent to the similarity reduction 2) (c) in 2.1.

3.2 $T=0$. Similar to the discussion for the Boussinesq equation in Ref.[4], we may let $X=1$. Then it is easy to find that the determining equation reads

$$U_t+U_x+U_{xx}+2U_{xu}U+U_{uu}U^2+h'U^2+hU_x=0. \tag{16}$$

3.2a. Taking $h=\lambda u, U=\dfrac{u}{x-t}$ in Eq.(16), then it is equivalent to the similarity reduction 1) in 2.2.

3.2b. Taking $h=-\dfrac{1}{a_0}\ln(u+k), U=\dfrac{a_0(u+k)}{t}$, then it admits the similarity reduction 2) in 2.2.

3.2c. $h(u)$ is presented in Eq.(13), let $U=\dfrac{h}{x-t}\mathrm{e}^{-\frac{a_0}{2}h^2}$, which corresponds to the similarity reduction 3) in 2.2.

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