Vol 11 No 4, April 2002
1009-1963/2002/11(04)/0313-06

Chinese Physics

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Study of the Lie symmetries of a relativistic variable mass system

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(Received 17 September 2001; revised manuscript received 28 November 2001)

The differential equations of motion of a relativistic variable mass system are given. By using the invariance of the differential equations under the infinitesimal transformations of groups, the determining equations and the restriction equations of the Lie symmetries of a relativistic variable mass system are built, and the structure equation and the conserved quantity of the Lie symmetries are obtained. Then the inverse problem of the Lie symmetries is studied. The corresponding Lie symmetries are found according to a known conserved quantity. An example is given to illustrate the application of the result.

Keywords: relativity, analytical mechanics, variable mass, Lie symmetries, conserved quantity
PACC: 0330

1. Introduction

There is a close relation between the symmetry and the conserved quantity in a mechanical system. The study of the theory and application of the symmetry and conserved quantity of a mechanical system is an important topic in mathematics, mechanics and physics. Modern methods to find conserved quantities of a mechanical system are mainly the Noether method[1] and the Lie method.[2] In the past twenty years, through the study of the symmetry and conserved quantity of a classical mechanical system, a series of important results has been obtained.[3−18] In this paper, we study the Lie symmetry and conserved quantity of a relativistic variable mass system. Firstly, the differential equations of motion of a relativistic variable mass system are built. Secondly, two kinds of problem for the Lie symmetry and the conserved quantity of the relativistic variable mass system are studied: one is to find the corresponding conserved quantity according to a given Lie symmetry, and the other is to find the corresponding Lie symmetry according to a known conserved quantity. Finally, an example is given to illustrate the application of the result.

2. D'Alembert–Lagrange principle of a relativistic variable mass system

We consider a mechanical system which consists of N variable mass particles. We suppose that, at time t, one particle whose rest mass is m_{oi} moves with a velocity $\boldsymbol{v}_i$, and another small particle whose rest mass is Δm_{oi} moves with a velocity $\boldsymbol{u}_i$. These combine with each other during the time interval Δt after t. Then the relativistic fundamental dynamical equation is[19]

$$m_{oi}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{v}_i}{\sqrt{1-v_i^2/c^2}}\right)=\boldsymbol{F}+\boldsymbol{\Phi}_i^*,$$

$$(i=1,2,\cdots,N), \tag{1}$$

where m_{oi} is a variable, and $\boldsymbol{\Phi}_i^*=\dot{m}_{oi}\left(\frac{\boldsymbol{u}_i}{\sqrt{1-u_i^2/c^2}}-\frac{\boldsymbol{v}_i}{\sqrt{1-v_i^2/c^2}}\right)$ is the relativistic reacting thrust.

We suppose that the configuration of a mechanical system is determined by n generalized coordinates $q_s(s=1,2,\cdots,n)$. Generally, $m_{oi}=m_{oi}(\boldsymbol{q},\dot{\boldsymbol{q}},\boldsymbol{t})$,[10] and we construct the generalized kinetic energy of the relativistic variable mass system as

$$T^*=\sum_{i=1}^{n}m_{oi}c^2(1-\sqrt{1-\dot{\boldsymbol{r}}_i^2/c^2}), \tag{2}$$

where $\dot{\vec{r}}_i=\vec{v}_i$. Then the d'Alembert–Lagrange principle of the relativistic variable mass system is expressed as[20]

$$\sum_{s=1}^{n}\left[-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial\dot{q}_s}+\frac{\partial T^*}{\partial q_s}+Q_s+P_s^*\right]\delta q_s=0, \tag{3}$$

where

$$P_s^* = \sum_{i=1}^{N}\left[\boldsymbol{\Phi}_s^* + \frac{m_{oi}\dot{\boldsymbol{r}}_i}{\sqrt{1-\dot{\boldsymbol{r}}_i^2/c^2}}\right]\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s} - \sum_{i=1}^{N}\frac{\partial m_{oi}}{\partial q_s}(1-\sqrt{1-\dot{\vec{r}}_i^2/c^2}) + \sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial m_{oi}}{\partial \dot{q}_s}c^2(1-\sqrt{1-\dot{\boldsymbol{r}}_i^2/c^2})\right]. \quad (4)$$

3. Dynamical equation of a relativistic variable mass system

3.1. Differential equation of motion of a relativistic variable mass holonomic system

We suppose that the system is only subjected to k holonomic constraints. Taking $n = 3N-k$ generalized coordinates $q_s (s = 1, 2, \cdots, n)$, then all δq_s are independent. So we can obtain the Lagrange equations of the relativistic variable mass holonomic system from Eq.(3) as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T^*}{\partial \dot{q}_s} - \frac{\partial T^*}{\partial q_s} = Q_s + P_s^* \quad (s = 1, 2, \cdots, n). \quad (5)$$

If the generalized forces Q_s can be divided into Q_s' with potential energy and Q_s'' without potential energy, then $Q_s = -\dfrac{\partial V}{\partial q_s} + Q_s''$, and Eq.(5) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} = Q_s'' + P_s^* \quad (s = 1, 2, \cdots, n) \quad (6)$$

where $L^* = T^* - V$ is the generalized Lagrangian of the relativistic variable mass system.

Extending Eqs.(6), we can obtain all the generalized accelerations[10]

$$\ddot{q}_s = h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, 2, \cdots, n). \quad (7)$$

3.2. Differential equation of motion of a relativistic variable mass nonholonomic system

We suppose that the system is subjected to g nonholonomic constraints as follows

$$f_\beta(t, q_s, \dot{q}_s) = 0 \quad (\beta = 1, 2, \cdots, g; s = 1, 2, \cdots, n), \quad (8)$$

which satisfy the Appell–Chetaev definition

$$\sum_{s=1}^{n}\frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0 \qquad (\beta = 1, 2, \cdots, g). \quad (9)$$

By using the Lagrange multiplier we can obtain Routh equations of the relativistic variable mass nonholonomic system from the principle (3) and the constraint conditions (9)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} = Q_s'' + P_s^* + \sum_{\beta=1}^{g}\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s} \quad (s = 1, 2, \cdots, n), \quad (10)$$

where λ_β is the constraint multiplier.

Before integrating the differential equation of motion, we can obtain λ_β, a function of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, from Eqs.(8) and (10). Substituting λ_β into Eqs.(10) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} = Q_s'' + P_s^* + \Lambda_s \quad (s = 1, 2, \cdots, n), \quad (11)$$

where

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{\beta=1}^{g}\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}. \quad (12)$$

Equations (11) are called the equations of motion of the corresponding relativistic variable mass holonomic system of the relativistic variable mass nonholonomic system expressed by Eqs.(8) and (10). Extending Eqs.(11), we can obtain all the generalized accelerations[10]

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \qquad (s = 1, 2, \cdots, n). \quad (13)$$

4. Determining equations of a relativistic variable mass system

4.1. Determining equations of a relativistic variable mass holonomic system

We choose the infinitesimal transformations of time and generalized coordinates as

$$t^* = t + \Delta t, \qquad q_s^* = q_s + \Delta q_s, \quad (14)$$

or their expansion formulae

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad q_s^* = q_s + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (15)$$

where ε is the infinitesimal parameter, and ξ_0 and ξ_s are the generators of infinitesimal transformations.

Then we introduce the generator vector for infinitesimal transformations

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \sum_{s=1}^{n} \xi_s \frac{\partial}{\partial q_s}, \tag{16}$$

and its first extended form

$$X^{(1)} = X^{(0)} + \sum_{s=1}^{n} (\dot{\xi}_s - \dot{q}_s \dot{\xi}_0) \frac{\partial}{\partial \dot{q}_s}, \tag{17}$$

and its second extended form

$$X^{(2)} = X^{(1)} + \sum_{s=1}^{n} (\ddot{\xi}_s - 2\ddot{q}_s \dot{\xi}_s - \dot{q}_s \ddot{\xi}_0) \frac{\partial}{\partial \ddot{q}_s}. \tag{18}$$

According to the invariant criterion of the differential equation, the invariance of the differential equations of motion (7) under the infinitesimal transformations (15) can be written as

$$X^{(2)}(\ddot{q}_s - h_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) = 0. \tag{19}$$

We can obtain the determining equations by expanding Eqs.(19)

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 h_s = X^{(1)}(h_s). \tag{20}$$

If generators ξ_0 and ξ_s satisfy Eqs.(20), then the corresponding transformations are called Lie symmetry transformations of the relativistic variable mass holonomic system.

4.2. Determining equations of a relativistic variable mass nonholonomic system

For a relativistic variable mass nonholonomic system, similar to the study of the relativistic variable mass holonomic system, we can find that the invariance condition of Eqs.(13) under infinitesimal transformations (15) can be expressed by the following determining equations

$$\ddot{\xi}_s - \dot{q}_s \ddot{\xi}_0 - 2\dot{\xi}_0 \alpha_s = X^{(1)}(\alpha_s). \tag{21}$$

If generators ξ_0 and ξ_s satisfy Eqs.(21), then the corresponding transformations are called Lie symmetry transformations of the corresponding relativistic variable mass holonomic system (11).

The invariance of constraint equations (8) under infinitesimal transformations (15) can be expressed by the following restriction equations

$$X^{(1)}[f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})] = 0. \tag{22}$$

If generators ξ_0 and ξ_s satisfy Eqs.(21) and (22), the corresponding transformations are called Lie symmetry transformations for the relativistic variable mass nonholonomic system expressed by Eqs.(8)and(10).

5. Structure equation and conserved quantity of a relativistic variable mass system

5.1. Structure equation and conserved quantity of a relativistic variable mass holonomic system

Proposition 1 For infinitesimal generators ξ_0 and ξ_s satisfying determining equations (20), if there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following equation

$$X^{(1)}(L^*) + L^* \dot{\xi}_0 + \sum_{s=1}^{n} (Q''_s + P^*_s)(\xi_s - \dot{q}_s \xi_0) + \dot{G} = 0, \tag{23}$$

then the relativistic variable mass holonomic system expressed by Eqs.(6) has a conserved quantity as follows

$$I = L^* \xi_0 + \sum_{s=1}^{n} (\xi_s - \dot{q}_s \xi_0) \frac{\partial L^*}{\partial \dot{q}_s} + G = \text{const}. \tag{24}$$

Proof

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} =& \dot{L}^* \xi_0 + L^* \dot{\xi}_0 + \sum_{s=1}^{n} \frac{\partial L^*}{\partial \dot{q}_s} (\dot{\xi}_s - \ddot{q}_s \xi_0 - \dot{q}_s \dot{\xi}_0) \\ &+ \sum_{s=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L^*}{\partial \dot{q}_s} (\xi_s - \dot{q}_s \xi_0) - X^{(1)}(L^*) - L^* \dot{\xi}_0 \\ &- \sum_{s=1}^{n} (Q''_s + P^*_s)(\xi_s - \dot{q}_s \xi_0) \\ =& \sum_{s=1}^{n} (\xi_s - \dot{q}_s \xi_0) \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} - Q''_s - P^*_s \right) = 0. \end{aligned}$$

Equation (23) is called the structure equation of Lie symmetries of the relativistic variable mass holonomic system (6).

5.2. Structure equation and conserved quantity of a relativistic variable mass nonholonomic system

For the system expressed by Eqs.(11), we have

Proposition 2 For infinitesimal generators ξ_0 and ξ_s satisfying determining equations (21), if there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the following equation

$$X^{(1)}(L^*) + L^* \dot{\xi}_0 + \sum_{s=1}^{n} (Q''_s + P^*_s + \Lambda_s)(\xi_s - \dot{q}_s \xi_0) + \dot{G} = 0, \tag{25}$$

then the corresponding relativistic variable mass holonomic system (11) has the conserved quantity in the form of Eq.(24).

The proof is similar to that for proposition 1. Equation (25) is called the structure equation of the corresponding relativistic variable mass holonomic system (11).

The conserved quantity of Lie symmetries of the relativistic variable mass nonholonomic system can be obtained from the conserved quantity of Lie symmetries of the corresponding relativistic variable mass holonomic system, as long as the restriction of nonholonomic constraints to infinitesimal generators is considered.

Proposition 3 For infinitesimal generators ξ_0 and ξ_s satisfying determining equations (21) and restriction equations (22), if there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfying the structure equation (25), then the relativistic variable mass nonholonomic system expressed by Eqs.(8) and (10) has the conserved quantity in the form of Eq.(24).

6. Lie symmetry inverse problem of a relativistic variable mass system

Supposing that a relativistic variable mass system has the integral

$$I = I(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \text{const}, \tag{26}$$

we have

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \sum_{s=1}^{n} \frac{\partial I}{\partial q_s}\dot{q}_s + \sum_{s=1}^{n} \frac{\partial I}{\partial \dot{q}_s}\ddot{q}_s = 0. \tag{27}$$

Multiplying Eqs.(11) by

$$\overline{\xi}_s = \xi_s - \dot{q}_s\xi_0, \tag{28}$$

and taking summation over s, we have

$$\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} - Q_s'' - P_s^* - \Lambda_s\right)\overline{\xi}_s = 0. \tag{29}$$

According to Eq.(4), P_s^* is generally a linear function of $\ddot{\boldsymbol{q}}$ and can be written as

$$P_s^* = \sum_{k=1}^{n} W_{sk}^*(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\ddot{q}_k + W_s^*(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{30}$$

where

$$W_{sk}^* = \sum_{i=1}^{N}\left[\frac{\partial m_{oi}}{\partial \dot{q}_k}\frac{\boldsymbol{u}_i}{\sqrt{1-\boldsymbol{u}_i^2/c^2}} \times \frac{\partial \boldsymbol{r}_i}{\partial q_s} + \frac{\partial^2 m_{oi}}{\partial \dot{q}_s \partial \dot{q}_k}c^2\left(1-\sqrt{1-\dot{\boldsymbol{r}}_i^2/c^2}\right) + \frac{\partial m_{oi}}{\partial \dot{q}_s}\frac{\dot{\boldsymbol{r}}_i}{\sqrt{1-\dot{\boldsymbol{r}}_i^2/c^2}}\cdot\frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_k}\right]. \tag{31}$$

Adding Eq.(27) to Eq.(29), and letting the coefficients of $\ddot{q}_k$ equal zero, we obtain

$$\sum_{s=1}^{n}\left(\frac{\partial^2 L^*}{\partial \dot{q}_s \partial \dot{q}_k} - W_{sk}^*\right)\overline{\xi}_s - \frac{\partial I}{\partial \dot{q}_k} = 0. \tag{32}$$

Supposing

$$\det(\omega_{sk}) = \det\left(\frac{\partial^2 L^*}{\partial \dot{q}_s \partial \dot{q}_k} - W_{sk}^*\right) \neq 0, \tag{33}$$

then from Eq.(32), we obtain

$$\overline{\xi}_s = \sum_{k=1}^{n} \tilde{\omega}_{sk}\frac{\partial I}{\partial \dot{q}_k}, \tag{34}$$

in which

$$\tilde{\omega}_{sk}\omega_{kl} = \delta_{sl}, \qquad \omega_{sk} = \frac{\partial^2 L^*}{\partial \dot{q}_s \partial \dot{q}_k} - W_{sk}^*. \tag{35}$$

Now letting integral (26) equal the conserved quantity (24), we have

$$I = L^*\xi_0 + \sum_{s=1}^{n}(\xi_s - \dot{q}_s\xi_0)\frac{\partial L^*}{\partial \dot{q}_s} + G. \tag{36}$$

Thus, given the gauge function G, we can obtain the infinitesimal generators ξ_0 and ξ_s from Eqs.(34) and (36).

Proposition 4 If generators ξ_0 and ξ_s determined by Eqs.(34) and (36) satisfy determining equations (21), then the acquired symmetries are the Lie symmetries of the corresponding relativistic variable mass holonomic system (11). If generators ξ_0 and ξ_s satisfy restriction equations (22) as well, then the symmetries are the Lie symmetries of the relativistic variable mass nonholonomic system expressed by Eqs.(8) and (10).

For a relativistic variable mass holonomic system, we only need to replace Eq.(29) by

$$\sum_{s=1}^{n}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} - Q_s'' - P_s^*\right)\overline{\xi}_s = 0. \tag{37}$$

Proposition 5 If generators ξ_0 and ξ_s determined by Eqs.(34) and (36) satisfy determining equations (20), then the obtained symmetries are the Lie

symmetries of the relativistic variable mass holonomic system (6).

7. Illustrative example

The Lagrangian of a relativistic variable mass system is

$$L^* = ke^{-\alpha t}c^2(1-\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}) - V,$$

$$(k, \alpha \text{ are constant}), \tag{38}$$

and the nonholonomic constraint is

$$f = \dot{q}_2 - t\dot{q}_1 = 0, \tag{39}$$

thus the non-potential generalized forces are

$$Q_1'' = -\frac{ke^{-\alpha t}}{t(1+t^2)}\frac{\dot{q}_1}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}},$$

$$Q_2'' = -\frac{ke^{-\alpha t}}{t(1+t^2)}\frac{\dot{q}_2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}},$$

while the potential energy V is constant, $u = 0$. We study the Lie symmetries and conserved quantities of the system.

Firstly, let us study the direct problem of the Lie symmetry. Equations (11) give

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[ke^{-\alpha t}\frac{\dot{q}_1}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}\right]\\ =&-\frac{ke^{-\alpha t}}{t(1+t^2)}\frac{\dot{q}_1}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}-\lambda t,\end{aligned}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[ke^{-\alpha t}\frac{\dot{q}_2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}\right]\\ =&-\frac{ke^{-\alpha t}}{t(1+t^2)}\frac{\dot{q}_2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}-\lambda.\end{aligned} \tag{40}$$

From Eqs.(39) and (40), we obtain

$$\lambda = \frac{ke^{-\alpha t}}{(1+t^2)}\frac{\dot{q}_1}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}. \tag{41}$$

Substituting formula (41) into Eqs.(40) yields

$$\begin{aligned}\ddot{q}_1 =&\dot{q}_1(1-(\dot{q}_1^2+\dot{q}_2^2)/c^2)\left(\alpha-\frac{1}{t}\right)-\dot{q}_1^2\dot{q}_2/c^2=\alpha_1,\\ \ddot{q}_2 =&\alpha\dot{q}_2(1-(\dot{q}_1^2+\dot{q}_2^2)/c^2)+\frac{\dot{q}_1^2\dot{q}_2}{c^2t}=\alpha_2.\end{aligned} \tag{42}$$

The determining equations (21) give

$$\begin{aligned}&\ddot{\xi}_1-\dot{q}_1\ddot{\xi}_0-2\dot{\xi}_0\alpha_1=X^{(1)}(\alpha_1),\\ &\ddot{\xi}_2-\dot{q}_2\ddot{\xi}_0-2\dot{\xi}_0\alpha_2=X^{(1)}(\alpha_2).\end{aligned} \tag{43}$$

Equations (43) have the following solutions

$$\xi_0=0,\quad \xi_1=1,\quad \xi_2=0, \tag{44}$$

$$\xi_0=0,\quad \xi_1=0,\quad \xi_2=1, \tag{45}$$

$$\xi_0=0,\quad \xi_1=1,\quad \xi_2=1. \tag{46}$$

So the generators correspond to Lie symmetries of the corresponding relativistic variable mass holonomic system.

The restriction equations (22) give

$$\dot{\xi}_2-\dot{q}_2\dot{\xi}_0-t(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)-\dot{q}_1\xi_0=0. \tag{47}$$

Because generators (44), (45) and (46) satisfy Eq.(47), they correspond to the Lie symmetries of the relativistic variable mass nonholonomic system.

For generators (45), from the structure equation (25) we can obtain the gauge function

$$G = c_0 \quad (\text{constant}). \tag{48}$$

From Eq.(24), we have the conserved quantity

$$I = ke^{-\alpha t}\frac{\dot{q}_2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}+c_0=\text{const.} \tag{49}$$

For generators (44) and (46), we cannot obtain the corresponding gauge function, so we cannot obtain the corresponding conserved quantity.

Secondly, let us study the inverse problem. We try to obtain Lie symmetries from the integral (49) of the system. We have

$$\frac{\partial I}{\partial\dot{q}_1}=ke^{-\alpha t}\frac{\dot{q}_1\dot{q}_2}{c^2[1-(\dot{q}_1^2+\dot{q}_2^2)/c^2]^{3/2}}, \tag{50}$$

$$\begin{aligned}\frac{\partial I}{\partial\dot{q}_2}=&ke^{-\alpha t}\left[\frac{1}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}\right.\\ &\left.+\frac{\dot{q}_2^2}{c^2[1-(\dot{q}_1^2+\dot{q}_2^2)/c^2]^{3/2}}\right].\end{aligned} \tag{51}$$

$$\begin{aligned}\omega_{11}=&\frac{\partial^2L^*}{\partial\dot{q}_1^2}\\ =&ke^{-\alpha t}\left[\frac{1}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}\right.\\ &\left.+\frac{\dot{q}_1^2}{c^2[1-(\dot{q}_1^2+\dot{q}_2^2)/c^2]^{3/2}}\right],\end{aligned} \tag{52}$$

$$\omega_{12}=\frac{\partial^2L^*}{\partial\dot{q}_1\partial\dot{q}_2}=\frac{\partial I}{\partial\dot{q}_1}, \tag{53}$$

$$\omega_{21}=\frac{\partial^2L^*}{\partial\dot{q}_2\partial\dot{q}_1}=\frac{\partial I}{\partial\dot{q}_1}=\omega_{12}, \tag{54}$$

$$\omega_{22} = \frac{\partial^2 L^*}{\partial \dot{q}_2^2} = \frac{\partial I}{\partial \dot{q}_2}. \tag{55}$$

From Eqs.(34), we obtain

$$\begin{aligned}\tilde{\xi}_1 =& \xi_1 - \dot{q}_1 \xi_0 \\ =& \tilde{\omega}_{11} \frac{\partial I}{\partial \dot{q}_1} + \tilde{\omega}_{12} \frac{\partial I}{\partial \dot{q}_2} \\ =& \tilde{\omega}_{11}\omega_{12} + \tilde{\omega}_{12}\omega_{22} = 0,\end{aligned} \tag{56}$$

$$\begin{aligned}\tilde{\xi}_2 =& \xi_2 - \dot{q}_2 \xi_0 \\ =& \tilde{\omega}_{21} \frac{\partial I}{\partial \dot{q}_1} + \tilde{\omega}_{22} \frac{\partial I}{\partial \dot{q}_2} \\ =& \tilde{\omega}_{21}\omega_{12} + \tilde{\omega}_{22}\omega_{22} = 1\end{aligned} \tag{57}$$

and Eq.(36) gives

$$\begin{aligned}& ke^{-\alpha t}\left[\frac{\dot{q}_1\xi_1 + \dot{q}_2\xi_2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}\right. \\ & + c^2\left(1 - \sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}\right)\xi_0 \\ & \left. - \frac{\dot{q}_1^2+\dot{q}_2^2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}}\xi_0\right] - V\xi_0 + G \\ =& ke^{-\alpha t}\frac{\dot{q}_2}{\sqrt{1-(\dot{q}_1^2+\dot{q}_2^2)/c^2}} + c_0.\end{aligned} \tag{58}$$

If we take $G = c_0$, then from Eqs.(56), (57) and (58) we can obtain

$$\xi_0 = 0, \quad \xi_1 = 0, \quad \xi_2 = 1. \tag{59}$$

8. Discussion

The results of this paper are of universal significance. They can be applied under relativistic and classical conditions to variable mass and constant mass systems. When m_{oi} is constant, then $P_s^* = 0$, and the results of this paper reduce to those of relativistic constant mass systems. When $v_i \ll c, u_i \ll c$, then T^*, $\boldsymbol{\Phi}_i^*$. P_s^* of the relativistic condition become $T = \sum_i^N \frac{1}{2} m_{oi}\dot{\boldsymbol{r}}_i^2$, $\boldsymbol{\Phi}_i = \frac{\mathrm{d}m_{oi}}{\mathrm{d}t}(\boldsymbol{u}_i - \boldsymbol{v}_i)$, $P_s = \sum_{i=1}^N \left[(\boldsymbol{\Phi}_i + \dot{m}_{oi}\dot{\boldsymbol{r}}_i)\cdot\frac{\partial \boldsymbol{r}_i}{\partial q_s} - \frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_{oi}}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\boldsymbol{r}}_i\cdot\dot{\boldsymbol{r}}_i\frac{\partial m_{oi}}{\partial \dot{q}_s}\right)\right]$ [10] under the classical condition, and the results of this paper become those of the classical variable mass system. When $v_i \ll c, u_i \ll c$ and m_{oi} is constant, the results of this paper can be turned into those of the classical constant mass system.

References

[1] Noether A E 1918 *Math. Phys.* KI **II** 235
[2] Lutzky M 1979 *J. Phys. A: Math. Gen.* **12** 973
[3] Vujanovic B 1986 *Acta Mech.* **65** 63
[4] Zhang J F 1989 *Chin. Sci. Bull.* **34** 1756 (in Chinese)
[5] Liu D 1990 *Sci. China* A **20** 1189 (in Chinese)
[6] Mei F X 1993 *Sci. China* A **36** 1456
[7] Zhao Y Y and Mei F X 1993 *Adv. Mech.* **23** 360 (in Chinese)
[8] Mei F X, Wu R H and Zhang Y F 1998 *Acta Mech. Sin.* **30** 468 (in Chinese)
[9] Mei F X 1999 *Appl. Math. Mech.* **20** 629
[10] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)
[11] Zhang Y and Mei F X 2000 *Chin. Sci. Bull.* **45** 1354
[12] Fang J H 2000 *Appl. Math. Mech.* **21** 836
[13] Zhang Y, Shang M and Mei F X 2000 *Chin. Phys.* **9** 401
[14] Mei F X 2000 *Acta Phys. Sin.* **49** 1207 (in Chinese)
[15] Mei F X and Shang M 2000 *Acta Phys. Sin.* **49** 1901 (in Chinese)
[16] Qiao Y F, Li R J and Zhao S H 2001 *Acta Phys. Sin.* **50** 811 (in Chinese)
[17] Zhang Y and Xue Y 2001 *Acta Phys. Sin.* **50** 816 (in Chinese)
[18] Zhang R C, Chen X W and Mei F X 2001 *Chin. Phys.* **10** 12
[19] Fang J H 1999 *Acta Phys. Sin.* **48** 1389 (in Chinese)
[20] Fang J H 2001 *Acta Phys. Sin.* **50** 1001 (in Chinese)