Vol 11 No 3, March 2002
1009-1963/2002/11(03)/0233-05
Chinese Physics
© 2002 Chin. Phys. Soc.
and IOP Publishing Ltd

Tracking control and synchronization of chaotic systems based upon sampled-data feedback*

Chen Shi-Hua(陈士华)†, Liu Jie(刘 杰),
Xie Jin(谢 进), and Lu Jun-An(陆君安)

School of Mathematical Science, Wuhan University, Wuhan 430072, China

(Received 5 September 2001; revised manuscript received 2 October 2001)

A novel tracking control and synchronization method is proposed based upon sampled-data feedback. This method can make a chaotic system approach any desired smooth orbit and synchronize the driving system and the response system, both in the same structure and in diverse structures. Finally, a numerical simulation with a Lorenz system is provided for the purpose of illustration and verification.

Keywords: synchronization, chaos, Lorenz system
PACC: 0545

1. Introduction

Chaos is a very interesting phenomenon in nonlinear systems, which has become a very active research area and has been intensively studied during the last three decades. A remarkable characteristic of chaos is that it is very sensitive to initial conditions, which makes any two adjacent orbits separate at a exponent rate eventually.[1] It seems impossible for two chaotic systems with the same structure to be synchronous even if they have the same initial condition because of noises and errors. For a long time, the opinion that chaotic dynamical systems cannot be synchronous has prevailed in the scientific community, and therefore scientific workers avoided studying chaos synchronization. But, in 1990, Pecora and Carroll[2] (P–C) found a control method to successfully keep two chaotic systems synchronous. Ever since Pecora and Carroll's original research work, there has been considerable work on chaos synchronization and its practical applications in secure communication, information science, biotic science and so on.

Nowadays, different techniques and methods have been put forward to achieve chaos synchronization, for instance, the P–C synchronization method,[2] linear and nonlinear feedback synchronization method,[3,4] impulsive synchronization method,[5] among many others.[6] The P–C synchronization method is valid only if the response system's greatest conditional Lyapunov exponent is less than zero. In using the feedback synchronization method one must analyse the system's dynamical behaviour in order to determine the convergence region, while in using the impulsive synchronization method one needs to construct a comparison system by the Lyapunov function and study the comparison system's stability. In this paper, we provide a novel synchronization method by synthesizing the entrainment control method[7] and sampled-data control method.[8] The suggested method synchronizes the response system and the driving system whether the driving system has the same structure or diverse structure as the response system. In the end, a computer simulation is also given for the purpose of illustration and verification.

2. Tracking control and synchronization method

2.1. Sampled-data feedback method

In this section, we denote a chaotic dynamical system as

$$\dot{x} = f(x) \tag{1}$$

where $x \in R^n$ is the state vector, $f : R^n \to R^n$ is a nonlinear smooth map, and $f(0) = 0$.

The control signals of the sampled-data feedback control method are constructed by the sampling sequence of the output of the chaotic system. The con-

*Project supported by the National Natural Science Foundation of China (Grant No 19531070).
†E-mail: shcheng@wuhee.edu.cn

trolled chaotic system using sampled values is given by

$$\begin{cases} \dot{x}(t) = f(x(t)) + Bu(k), & t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + Dx(k) & k = 0, 1, 2, \cdots \end{cases} \tag{2}$$

where $u \in R^n, B, C, D \in R^{n\times n}, t \in R_+, x(k)$ is the sampled value of $x(t)$ at $t = k\Delta$ and Δ is the sampling duration.

The controlled chaotic system is a kind of hybrid dynamical system, which contains both continuous time dynamics and discrete time dynamics, thus the stability of this kind of hybrid system should be considered in both the continuous time domain and the discrete time domain. Since $f(0) = 0$, $(x, u) = (0, 0)$ is an equilibrium point of the system (2).

Suppose $f \in C^1(R^n, R^n)$, then the asymptotic stability of the equilibrium point (0,0) can be determined from that of the trivial solution of the associated linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(k), & t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + Dx(k), & k = 0, 1, 2, \cdots \end{cases} \tag{3}$$

where $A \in R^{n\times n}$ is given by $A = \dfrac{\partial f(0)}{\partial x}$. The following theorem gives an explicit criterion for the stability of the controlled chaotic system.[9]

Theorem 1 Suppose $f \in C^1(R^n, R^n)$ and A is nonsingular, then the point $(x, u) = (0, 0)$ of the controlled chaotic system (2) is uniformly asymptotically stable if the following matrix

$$T = \begin{pmatrix} e^{A\Delta} & A^{-1}e^{A\Delta}B - A^{-1}B \\ D & C \end{pmatrix}$$

is schur stable, that is, all the eigenvalues of T lie in the unit disc centred at the origin.

2.2. Synchronization of chaotic systems

We consider the response system and the driving system described by the following state equations

$$\dot{x} = f(x), \tag{4}$$

$$\dot{y} = h(y), \tag{5}$$

where $x, y \in R^n$ are the state vectors, and $f, h : R^n \to R^n$ are nonlinear smooth maps. The controlled system is

$$\dot{x} = f(x) + U, \tag{6}$$

where $U \in R^n$ is a controller. The aim of chaos synchronization is to deign a controller U such that

$$\lim_{t\to+\infty} |x(t) - y(t)| = 0. \tag{7}$$

In order to achieve the chaos synchronization, we first design a controller, which enables the controlled system (6) to track any desired trajectory.

Theorem 2 Suppose $r(t) \in C^1(R_+, R^n), f \in C^1(R^n, R^n), A$ is nonsingular and T is schur stable, then the trajectory $x = r(t)$ of the controlled system

$$\begin{cases} \dot{x}(t) = f(x(t)) + \dot{r}(t) - f(r(t)) - g'(r(t))e(t) \\ \qquad +Bu(k), & t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + De(k), & k = 0, 1, 2, \cdots \end{cases} \tag{8}$$

is uniformly asymptotically stable. Here $e(t) = x(t) - r(t), e(k)$ is the sampled value of $e(t)$ at $t = k\Delta, g(x) = f(x) - Ax$, and A, B, C, D, T, Δ are same as in section 2.1.

Proof The controlled system (8) can be reduced to

$$\begin{cases} \dot{e}(t) = Ae(t) + g(x(t)) - g(r(t)) - g'(r(t))e(t) \\ \qquad +Bu(k), & t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + De(k), & k = 0, 1, 2, \cdots \end{cases} \tag{9}$$

Thus its associated linear system at $(e(t), u) = (0, 0)$ is

$$\begin{cases} \dot{e}(t) = Ae(t) + Bu(k), & t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + De(k), & k = 0, 1, 2, \cdots \end{cases} \tag{10}$$

because the following equation

$$\lim_{|e(t)|\to 0} \frac{|g(x(t)) - g(r(t)) - g'(r(t))e(t)|}{|e(t)|} = 0$$

is correct.

According to theorem 1, the equilibrium point $(e(t), u) = (0, 0)$ of the system (10) is uniformly asymptotically stable, which implies the theorem 2.

Theorem 2 gives a controller U so that the controlled system (6) tracks any desired trajectory $r(t)$, including stable or unstable limit cycles as well as chaotic trajectory. In practice, one can omit the term $g'(r(t))e(t)$ in Eq.(8) and simply synthesize the entrainment control method and sampled-data control method and design U as

$$U = \dot{r}(t) - f(r(t)) + Bu(k). \tag{11}$$

We fail to prove that the conclusion of theorem 2 is still correct with this kind of controller theoretically,

but the simulation below shows that the controller as in Eq.(11) is effective. Therefore, we have the following two lemmas.

Lemma 1 Suppose $f \in C^1(R^n, R^n)$ and A is nonsingular. T is schur stable, then the controlled system

$$\begin{cases} \dot{x}(t) = f(x(t)) + Bu(k), & t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + De(k), & k = 0, 1, 2, \cdots \end{cases}$$

and the driving system

$$\dot{y}(t) = f(y(t))$$

are uniformly asymptotic synchronized, where $e(t) = x(t) - y(t), e(k)$ is the sampled value of $e(t)$ at $t = k\Delta$ and A, B, C, D, T and Δ are same as in theorem 2.

Lemma 2 Suppose $f \in C^1(R^n, R^n)$ and A is nonsingular, T is schur stable, then the controlled system

$$\begin{cases} \dot{x}(t) = f(x(t)) + h(y(t)) - f(y(t)) + Bu(k), \\ \qquad\qquad t \in [k\Delta, (k+1)\Delta], \\ u(k+1) = Cu(k) + De(k), \quad k = 0, 1, 2, \cdots \end{cases} \tag{12}$$

and the drive system

$$\dot{y}(t) = h(y(t))$$

are uniformly asymptotic synchronized, where $e(t) = x(t) - y(t), e(k)$ is the sampled value of $e(t)$ at $t = k\Delta$ and A, B, C, D, T, Δ are same as in theorem 2.

3. Simulation study

In this section a computer simulation with the Lorenz system is given for the purpose of illustration and verification. The Lorenz equation is given by

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = (\mu - z)x - y, \\ \dot{z} = xy - bz, \end{cases}$$

where $x, y, z \in R$ are state variables and σ, μ and b are three real positive parameters. When these parameters are chosen as $\sigma = 10$, $b = \frac{8}{3}$, $\mu \geq 27.74$, the system has a chaotic attractor. For convenience, the parameter μ is selected to be 28. Sampling duration Δ=0.001, the control matrices are selected to be

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.966 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then A is nonsingular and

$$T = \begin{bmatrix} 0.9902 & 0.0099 & 0 & 0 & 0 & 0 \\ 0.0278 & 0.9991 & 0 & 0 & 0.0010 & 0 \\ 0 & 0 & 0.9973 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0.9960 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues and their norms of T are listed in Table 1.

Table 1. The eigenvalues of matrix T.

Real	Image	Norm
0.9992	0	0.9992
0.9780	0.0240	0.9783
0.9780	–0.0240	0.9783
0.9973	0	0.9973
0	0	0
0	0	0

Obviously T is schur stable.

3.1. Lorenz system tracking any desired trajectory

Suppose the reference signal is $r(t) = (\sin 2\pi t, \cos 2\pi t, 0)^T$, with the controller (11) applied, the Lorenz system tracks the trajectory $r(t)$ as expected. Figures 1 and 2 show the phase graph and the state vector graphs of the controlled Lorenz system respectively.

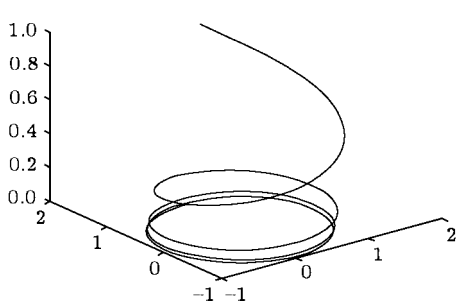


Fig.1. Phase graph of the controlled system.

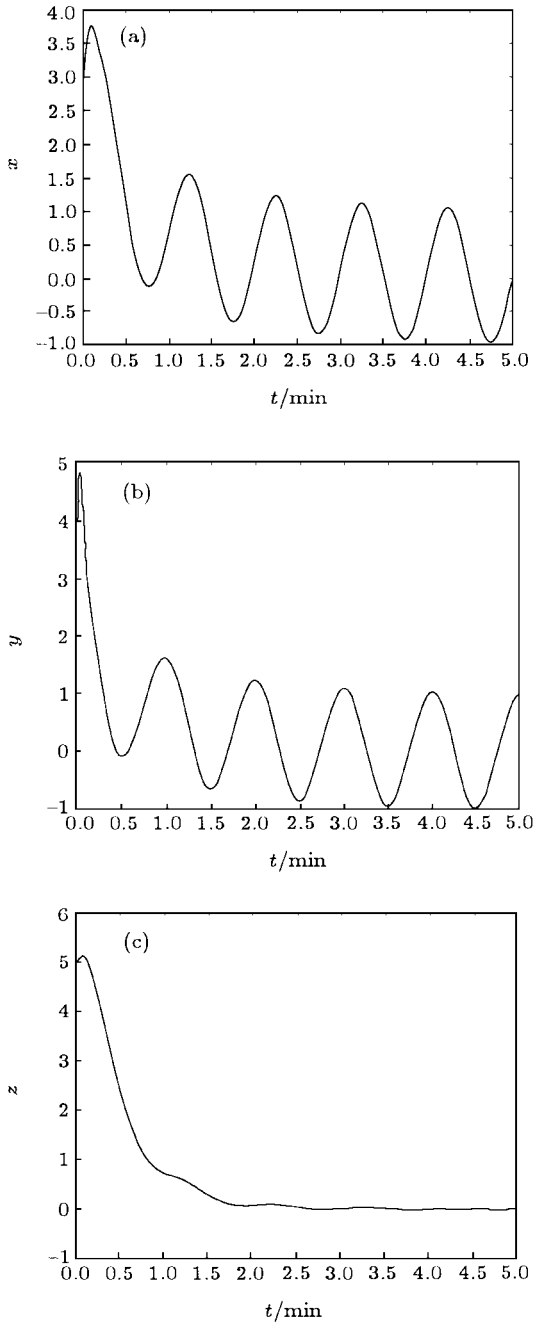


Fig.2. State vector graphs of the controlled system.

3.2. Chaos synchronization with the driving system and response system having the same structure

Suppose the driving system and the response system have the same structure, then the driving system is given by

$$\begin{aligned}\dot{\hat{x}} &= \sigma(\hat{y}-\hat{x}),\\ \dot{\hat{y}} &= (\mu-\hat{z})\hat{x}-\hat{y},\\ \dot{\hat{z}} &= \hat{x}\hat{y}-b\hat{z}.\end{aligned}$$

According to lemma 1 the controlled Lorenz system is

$$\begin{aligned}&\dot{x} = \sigma(y-x),\\ &\dot{y} = (\mu-z)x-y+u_2(k), \quad t\in[k\Delta,(k+1)\Delta],\\ &\dot{z} = xy-bz,\\ &u_2(k+1) = 0.966u_2(k)-e_2(k), \quad k=0,1,2\cdots\end{aligned}$$

where $e_2(t) = y(t)-\hat{y}(t)$ and $e_2(k)$ is the sampled value of $e_2(t)$ at $t=k\Delta$.

The initial states of the driving system and the controlled system are $(8.0,9.0,10.0)^{\mathrm{T}}$ and $(3.0,\ 4.0,\ 5.0)^{\mathrm{T}}$ respectively. We find that the synchronization occurs even though the controlled system and the driving system have different initial states. The error variables varying with time are shown in Fig.3.

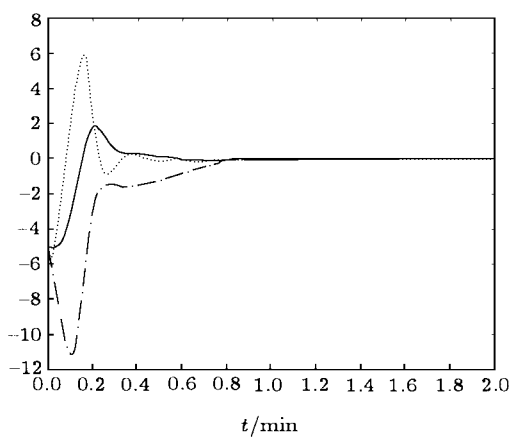


Fig.3. Graphs of synchronization errors: solid line, $e_1(t) = x(t)-\hat{x}(t)$; dotted line, $e_2(t) = y(t)-\hat{y}(t)$; dashed-dotted line, $e_3(t) = z(t)-\hat{z}(t)$.

3.3. Chaos synchronization with the driving system and response system having different structures

Suppose the driving system is given by

$$\begin{cases}\dot{\hat{x}} = 35(\hat{y}-\hat{x}),\\ \dot{\hat{y}} = -7\hat{x}-\hat{x}\hat{z}+28\hat{y},\\ \dot{\hat{z}} = \hat{x}\hat{y}-3\hat{z},\end{cases}$$

which has been found by Chen[10] recently in anti-controlling chaos. From a control engineering point of view, Chen's chaotic attractor is relatively difficult to control, as compared to the Lorenz system due to its rapid change in the z-direction.

The initial states of the driving system and the controlled Lorenz system as in Eq.(12) are (10.0, 9.0, 8.0)$^{\mathrm{T}}$ and (2.0, 4.0, 5.0)$^{\mathrm{T}}$ respectively. We find

that the synchronization occurs even though the driving and the response systems have different structures. The synchronization errors varying with time are shown is Fig.4.

4. Conclusion

We have provided a novel tracking control and synchronization method based on sampled-data feedback. This new and effective method can drive a chaotic system to approach any desired smooth orbit and synchronize the driving system and response system, both with the same structure and with diverse structures. The controller designed in this paper, especially in self-synchronization, uses the sampled value of output of the chaotic system to construct control signals, unlike most previous results where control inputs were constructed by the continuous observations of the outputs of the chaotic systems. Furthermore, a numerical simulation with Lorenz chaotic system is given to show the effectiveness and feasibility of the developed method.

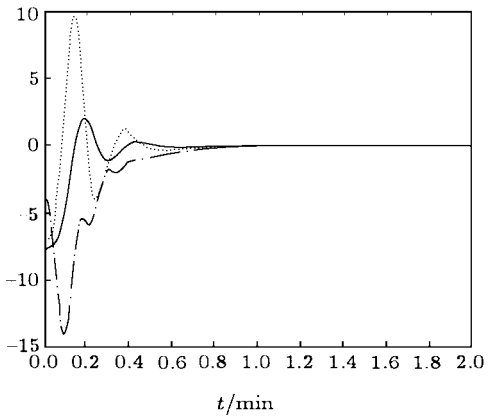


Fig.4. Graphs of synchronization errors: solid line, $e_1(t) = x(t) - \hat{x}(t)$; dotted line, $e_2(t) = y(t) - \hat{y}(t)$; dashed-dotted line, $e_3(t) = z(t) - \hat{z}(t)$.

References

[1] Chen Y X and Wang G R 1995 *Acta Phys. Sin.* **44** 1382 (in Chinese)
[2] Pecora L M and Carroll T L 1990 *Phys. Rev. Lett.* **64** 821
[3] Li L X, Peng H P, Lu H B and Guan X P 2001 *Chin. Phys.* **10** 796
[4] Wu W G and Gu T X 2000 *Acta Phys. Sin.* **49** 1922 (in Chinese)
[5] Liu F, Mu Z L and Qiu Z L 1999 *Acta Phys. Sin.* **48** 1198 (in Chinese)
[6] Liu F, Ren Y, Shan X M and Qiu Z L 2001 *Chin. Phys.* **10** 606
[7] Jackson E A 1991 *Physica* D **50** 341
[8] Yang L B and Yang T 2000 *Acta Phys. Sin.* **49** 1039 (in Chinese)
[9] Ye H, Michel A N and Hou L 1998 *IEEE Trans. Autom. Control* **43** 461
[10] Guan X P, Fan Z P, Peng H P and Wang Y Q 2001 *Chin. Phys.* **50** 11