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New ways of deriving Arnowitt–Deser–Misner constraint equations in four-dimensional gravity*

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In this paper, the Arnowitt–Deser–Misner (ADM) constraint equations are naturally derived in two different ways. One method is to construct a one-parametric gravitational action in the Lorentzian spacetime. Hence, the one-parametric ADM constraint equations can be obtained. The other method is to apply the double complex function method to Einstein–Hilbert gravitational fields in Hamiltonian formulation. Therefore the double ADM constraint equations can be obtained, in which the well-known ADM constraint equations are included as a special case.

Keywords: Arnowitt–Desner–Misner (ADM) constraint equations, Hamiltonian formulation, spacetime signature
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1. Introduction

It is well known that the Einstein gravitational equations in Hamiltonian formulation can be written as follows

$$-\sqrt{q}^{(3)}R+\frac{1}{\sqrt{q}}\left(\widetilde{p}^{ab}\widetilde{p}_{ab}-\frac{1}{2}\widetilde{p}^{2}\right)=0,$$

$$D_b\widetilde{p}^{ab}=0,$$

$$\dot{q}_{ab}=\frac{2N}{\sqrt{q}}\left(\widetilde{p}_{ab}-\frac{1}{2}q_{ab}\widetilde{p}+2D_{(a}N_{b)}\right),$$

$$\begin{aligned}\dot{\widetilde{p}^{ab}}=-\frac{\delta H_G}{\delta q_{ab}}=&\sqrt{q}N\left({}^{(3)}R^{ab}+\frac{1}{2}q^{ab(3)}R\right)\\&+\frac{N}{2\sqrt{q}}q^{ab}\left(\widetilde{p}_{cd}\widetilde{p}^{cd}-\frac{1}{2}\widetilde{p}^{2}\right)\\&-\frac{2N}{\sqrt{q}}\left(\widetilde{p}^{ac}\widetilde{p}^{b}_{c}-\frac{1}{2}\widetilde{p}\widetilde{p}^{ab}\right)\\&+\sqrt{q}\left(D^aD^bN-q^{ab}D^cD_cN\right)\\&+D_c\left(N^c\widetilde{p}^{ab}\right)-2\widetilde{p}^{c}\left({}^aD_cN^b\right).\end{aligned}\quad(1)$$

The first two equations in Eqs.(1) are the famous Arnowitt–Deser–Misner (ADM) constraint equations[1] with canonical conjugate variables, i.e., the 3-metric q_{ab} and its conjugate variables $\widetilde{p}^{ab}$. The last two equations in Eq.(1) are the evolution equations describing the change of dynamical variables q_{ab} and $\widetilde{p}^{ab}$ with time.

Recently, there have been many achievements in the study of ADM constraint equations.[2−6] Especially, Barbero[5,6] has constructed a gravitational action with two real parameters in the Euclidean spacetime (i.e. the spacetime signature is the Euclidean signature $\eta_E=diag(+1,+1,+1,+1)$), and has given the gravitational constraint equations containing two parameters. Moreover, by adjusting the values of the two parameters the well-known ADM constraint equations can be obtained. Thus, there naturally arises the problem whether we can construct a new gravitational action with only one parameter, by which the ADM constraint equations can be naturally derived. We answer this question affirmatively in this paper.

We find that if we construct a new gravitational action with one parameter in the Lorentzian spacetime (i.e. the spacetime signature is the Lorentzian signature $\eta_L=diag(-1,+1,+1,+1)$), we can not only obtain the familiar ADM constraint equations, but we can also control the spacetime signature by adjusting the value of this parameter. Hence, our method uses fewer parameters and has a simpler form in comparison with Barbero's method.[5]

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On the other hand, we know that the double complex function method (DCFM)[7] has been applied to many fields in physics, and has made some achievements.[8−13] In this paper we also use the DCFM to discuss the doubling of ADM constraint equations, which respectively correspond to Lorentzian and Euclidean gravity.

We begin with a brief introduction of the DCFM. Let J denote the double imaginary unit, i.e. $J = \mathrm{i}(\mathrm{i}^2 = -1)$ or $J = \epsilon(\epsilon^2 = +1,\ \epsilon \neq 1)$. $Z(J) = a(J) + Jb(J)$ is called a double complex number, where $a(J)$ and $b(J)$ are double real numbers. Sometimes $Z(J)$ may be directly written as $Z(J) = (Z_C, Z_H)$, where $Z_C = Z(J = \mathrm{i})$, $Z_H = Z(J = \epsilon)$, which are called ordinary complex and hyperbolic complex numbers, respectively.

2.One-parametric ADM constraint equations

In order to work out the new method of deriving ADM constraint equations, we construct a new gravitational action with one parameter, i.e. a one-parametric gravitational action, and decompose it in 3+1-form.

First, let us consider a four-dimensional manifold $M = \Sigma \times \boldsymbol{R}$, and introduce in M a metric g_{ab} with Lorentzian signature $\eta_L = diag(-1, +1, +1, +1)$. M can be foliated by three-dimensional surface defined by the constant value of a certain scaler function τ, where τ is a fixed scalar function monotonically growing in the $\boldsymbol{R}$ direction that can be interpreted as an external time variable (the time variable t is a special case as shown below). We can also introduce a unit covariant normal η^a, which is defined as

$$\eta_a = \frac{\partial_a \tau}{\left(g^{bc}\partial_b\tau\partial_c\tau\right)^{\frac{1}{2}}} \tag{2}$$

and satisfies

$$\eta^a = g^{ab}\eta_b,$$
$$\eta^a\eta_a = -1. \tag{3}$$

In addition, we define the three-dimensional metric q_{ab}, the lapse N, and the shift N^a via

$$q_{ab} := q_a^m q_b^n g_{mn} = g_{ab} + \eta_a\eta_b \quad \left(q_a^b = \delta_a^b + \eta_a\eta^b\right),$$
$$N := -\eta^a\tau^b g_{ab},$$
$$N^a := q_b^a\tau^b. \tag{4}$$

Note that in terms of N and N^a, we can write $\tau^a = N\tau^a + N^a$. Furthermore, they satisfy the relations

$$N^a\eta_a = 0,$$
$$q_{ab}\eta^b = 0. \tag{5}$$

According to the above, now we can construct a new one-parametric gravitational action as follows

$$S[g_{ab};\alpha] = \int_M \mathrm{d}^4x\sqrt{g}[2\alpha^2 G^{ab} + (1-\alpha^2)R^{ab}]\eta_a\eta_b, \tag{6}$$

where α is an introduced parameter, $G^{ab} = R^{ab} - \frac{1}{2}g^{ab}R$ $(a, b = 1, 2, 3, 4)$ are the four-dimensional Einstein tensors, and R^{ab} are Riemann curvature tensors.

The next step in deriving the one-parametric ADM constraint equations is to decompose the new action and write it in the form

$$S\left[g_{ab};\alpha\right] = \int_{\boldsymbol{R}} \mathrm{d}\tau L(q, q, \alpha), \tag{7}$$

where L will be the Lagrangian.

According to the usual procedure,[2,6] from the following differential geometric identities

$$G_{ab}\eta^a\eta^b = \frac{1}{2}\left({}^{(3)}R - K_{ab}K^{ab} + K^2\right),$$
$$R_{ab}\eta^a\eta^b = -K_{ab}K^{ab} + K^2 + \nabla_b\left(\eta^a\ \nabla_a\eta^b - \eta^b\ \nabla_a\eta^a\right), \tag{8}$$

and

$$\dot{q}_{ab} = \pounds_{\vec{\tau}} q_{ab} = 2NK_{ab} + 2D_{(a}N_{b)}$$

we can rewrite the action (6) by a complicated calculation

$$S\left[g_{ab};\alpha\right] = \int_{\boldsymbol{R}}\mathrm{d}\tau\int_\Sigma \mathrm{d}^3x\sqrt{q}N \times\left[\alpha^2{}^{(3)}R + (K^2 - K_{ab}K^{ab})\right], \tag{9}$$

where $K_{ab} = q_a^c q_b^d\ \nabla_c\eta_d = q_a^c\ \nabla_c\eta_b$ is the extrinsic curvature of the τ=const surfaces and ${}^{(3)}R$ is the scaler curvature of the unique, torsion-free spatial derivative operator D_a compatible with the induced metric q_{ab} . Hence, the Lagrangian can be directly written as

$$L = \int_\Sigma \mathrm{d}^3x\sqrt{q}N\left[\alpha^2{}^{(3)}R + (K^2 - K_{ab}K^{ab})\right]. \tag{10}$$

Here q and K are the determinants of q_{ab} and K_{ab}, respectively.

Furthermore, in terms of the definitions of the momenta $\widetilde{p}^{ab}$ canonically conjugate to q^{ab} and the Hamiltonian H

$$\widetilde{p}^{ab} = \frac{\delta L}{\delta\dot{q}^{ab}},$$
$$H = \widetilde{p}^{ab}\dot{q}_{ab} - L_G. \tag{11}$$

We can obtain the Hamiltonian by a lengthy calculation

$$H=\int_{\Sigma}\mathrm{d}^3x\Big\{N\Big[-\alpha^2\sqrt{q}^{(3)}R+\frac{1}{\sqrt{q}}\Big(\frac{1}{2}\widetilde{p}^2-\widetilde{p}^{ab}\widetilde{p}_{ab}\Big)\Big]-2N_aD_b\widetilde{p}^{ab}\Big\}, \quad (12)$$

where

$$\widetilde{p}^{ab}=\sqrt{q}\left(Kq^{ab}-K^{ab}\right). \quad (13)$$

Lastly, by means of the Dirac constraint analysis[14] the corresponding first class constraint equations can be obtained as follows

$$\alpha^2\sqrt{q}^{(3)}R+\frac{1}{\sqrt{q}}\left(\widetilde{p}^{ab}\widetilde{p}_{ab}-\frac{1}{2}\widetilde{p}^2\right)=0,$$

$$D_b\widetilde{p}^{ab}=0, \quad (14)$$

which are called the one-parametric ADM constraint equations corresponding to the action (6).

From Eqs.(14) it is easy to see that, when $\alpha=i$, Eqs.(14) become

$$-\sqrt{q}^{(3)}R+\frac{1}{\sqrt{q}}\left(\widetilde{p}^{ab}\widetilde{p}_{ab}-\frac{1}{2}\widetilde{p}^2\right)=0,$$

$$D_b\widetilde{p}^{ab}=0. \quad (15)$$

These are just the familiar ADM constraint equations corresponding to Lorentzian signature.[1] In this case the external time variable τ is just the time t of the spacetime variables. But when $\alpha=1,\epsilon$, Eqs.(14) become

$$\sqrt{q}^{(3)}R+\frac{1}{\sqrt{q}}\left(\widetilde{p}^{ab}\widetilde{p}_{ab}-\frac{1}{2}\widetilde{p}^2\right)=0,$$

$$D_b\widetilde{p}^{ab}=0. \quad (16)$$

These are exactly in agreement with Barbero's results,[5] i.e. the ADM constraint equations corresponding to Euclidean signature.

3. The double ADM constraint equations

In the following we apply the DCFM[6] to the gravitational field. We find, by means of the doubling of the spacetime 4-manifold M and the corresponding Einstein–Hilbert action $S[g_{ab}]=\int_M\sqrt{g}R$, that we can not only derive the ADM constraint equations but we can also in a unified manner describe Lorentzian and Euclidean gravity.

Let $M(J)$ be a doubling spacetime 4-manifold, i.e. $M(J=\mathrm{i})=M_L$ (with Lorentzian signature) and $M(J=\epsilon)=M_E$ (with Euclidean signature). Thus, the double spacetime metric $g_{ab}(J)$ on $M(J)$ can be expressed as

$$g_{ab}(J)=\begin{pmatrix}N_iN^i+J^2N^2 & N_i\\ N_k & q_{ik}\end{pmatrix}, \quad (17)$$

$$g^{ab}(J)=\begin{pmatrix}\dfrac{J^2}{N^2} & -\dfrac{J^2N^i}{N^2}\\ -\dfrac{J^2N^k}{N^2} & q^{ik}+\dfrac{J^2N^iN^k}{N^2}\end{pmatrix}. \quad (18)$$

When taking $J=\mathrm{i}$, $g_{ab}(J=\mathrm{i})=g_{ab}(L)$ is just compatible with the ADM decomposition of the metric with Lorentzian signature. When taking $J=\epsilon$, $g_{ab}(J=\epsilon)=g_{ab}(E)$ is the one of the metric with Euclidean signature.

With this decomposition of the double metric, the gravitational action in the double spacetime 4-manifold $M(J)$ can be written as

$$S_{EH}[g_{ab}(J)]=\int_{M(J)}\sqrt{-g(J)}\,R(J). \quad (19)$$

To decompose the action $S_{EH}[g_{ab}(J)]$ we need to introduce a double unit covariant normal $n^a(J)$, which is defined as

$$n_a(J)=\frac{\partial_a t}{\left(g^{bc}(J)\,\partial_b t\partial_c t\right)^{\frac{1}{2}}}$$

and satisfies

$$n^a(J)=g^{ab}(J)\,n_b(J),$$

$$n^a(J)\,n_a(J)=J^2. \quad (20)$$

The corresponding three-dimensional metric q_{ab}, the lapse $N(J)$ and the shift $N^a(J)$ are

$$q_{ab}:=g_{ab}(J)-J^2n_a(J)\,n_b(J)$$

$$\left(q_a^b=\delta_a^b-J^2n_a(J)\,n^b(J)\right),$$

$$N(J):=J^2n^a(J)\,n^b(J)\,g_{ab}$$

and

$$N^a(J):=q_a^b(J)\,t^b, \quad (21)$$

respectively. Thus, the double action $S_{EH}[g_{ab}(J)]$ can be rewritten as

$$S_{EH}[g_{ab}(J)]=\int\mathrm{d}t\int_{\Sigma}\mathrm{d}^3x\sqrt{q}N(J)\,R(J). \quad (22)$$

Furthermore, by means of the Gauss–Codazzi relations

$${}^{(3)}R^d_{abc}=q^e_aq^f_bq^g_cq^{d(4)}_hR^h_{efg}+K_{ac}K^d_b-K_{bc}K^d_a$$

and

$$D_a K^a_b - D_b K^a_a = R_{ac} q^a_b n^c, \tag{23}$$

where ${}^{(3)}R^d_{abc}$ are three-dimensional curvature tensors on the space-like submanifold Σ, and D_a is the spatial covariant derivative operator D_a satisfying $D_c q_{ab} = 0$, we can express $R(J)$ in Eq.(19) as

$$R(J) = -J^{2(3)}R + K_{ab}K^{ab} - K^2 + 2\nabla_b \left(n^a \nabla_a n^b - n^b \nabla_a n^a\right). \tag{24}$$

Putting Eq.(24) into the double action $S_{EH}[g_{ab}(J)]$ in Eq.(19), we can rewrite it by 3+1-decomposition as

$$S_{EH}[g_{ab}(J)] = \int \mathrm{d}t \int_\Sigma \mathrm{d}^3x \sqrt{q} N \times \left(-J^{2(3)}R + K_{ab}K^{ab} - K^2\right). \tag{25}$$

Thus, the corresponding Hamiltonian can be obtained by means of its definition in Eq.(11)

$$H_{EH}(J) = \int_\Sigma \mathrm{d}^3x \left\{ N \left[J^{2(3)}R + \widetilde{p}^{ab}_{(EH)} \widetilde{p}_{ab(EH)} - \frac{1}{2}\widetilde{p}^2_{(EH)} \right] - 2N_a D_b \widetilde{p}^{ab}_{(EH)} \right\}, \tag{26}$$

where

$$\widetilde{p}^{ab}_{(EH)} = \sqrt{q}\left(K^{ab} - Kq^{ab}\right). \tag{27}$$

Finally, in terms of the Dirac constraint analysis[14] the corresponding constraint equations can be naturally derived as follows

$$J^2\sqrt{q}^{(3)}R + \frac{1}{\sqrt{q}}\left(\widetilde{p}^{ab}_{(EH)}\widetilde{p}_{ab(EH)} - \frac{1}{2}\widetilde{p}^2_{(EH)}\right) = 0,$$

$$D_b \widetilde{p}^{ab}_{(EH)} = 0. \tag{28}$$

These are called the double ADM constraint equations. When $J = \mathrm{i}$, Eqs.(28) change into the familiar ADM constraint equations (15) describing Lorentzian gravity, and when $J = \epsilon$, Eqs.(28) change into Barbero's results, i.e. the constraint equations (16) describing Euclidean gravity. It follows that we once again obtain the ADM constraint equations by using the DCFM.

So far we have naturally derived the ADM constraint equations with Lorentzian and Euclidean signatures in two different ways: the first is to construct a new gravitational action, i.e. a one-parametric gravitational action, by which we obtain the one-parametric ADM constraint equations; and the second is to give a double Einstein–Hilbert action by using the DCFM, by which we obtain the double ADM constraint equations. Although the two ways are completely different, the same results can be obtained by adjusting the value of the parameter α. It is clear that there are internal relations between the two different methods. If the familiar ADM constraint equations (15) and (16) corresponding to the Lorentzian and Euclidean signatures are respectively denoted by ADM(η_L) and ADM(η_E), the one-parametric and double ADM constraint equations (14) and (28) are respectively denoted by ADM(α) and ADM(J). The above discussion can be shown in a diagram (see Fig.1).

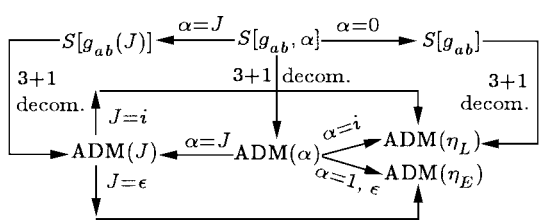


Fig.1.

It follows that the double ADM constraint equations are included as a special case of $\alpha = J$ in the one-parametric ADM equations.

References

[1] Arnowitt R, Deser S and Misner C W 1962 *Gravitation: An Introduction to Current Research* (Princeton, NJ: Princeton University Press)

[2] Wald R M 1984 *General Relativity* (Chicago: University of Chicaga Press)

[3] Wang Y J and Tang Z M 1990 *The Theory and Effect of Gravitational* (Changsha: Hu'nan Science and Technology Press)

[4] Romano J D 1993 *Gen. Relativ. Grav.* **25** 759

[5] Barbero G 1995 *Phys. Rev.* D **51** 5507

[6] Barbero G 1996 *Phys. Rev.* D **54** 1492

[7] Zhong Z Z 1985 *J. Math. Phys.* **26** 404
Zhong Z Z 1985 *J. Math. Phys.* **26** 2589

[8] Hucks J 1993 *J. Math. Phys.* **34** 5986

[9] Wu Y B and Gui Y X 1999 *Gen. Relativ. Grav.* **31** 165

[10] Wu Y B 1998 *Inter. J. Theor. Phys.* **37** 2127

[11] Wu Y B and Li J L 2001 *Chin. Phys. Lett.* **18** 328

[12] Wu Y B and Gui Y X 2001 *Chin. Phys.* **10** 902

[13] Li J L and Wu Y B 2001 *Acta Phys. Sin.* **50** 411 (in Chinese)

[14] Dirac P 1984 *Lectures on Quantum Mechanics* (New York: Belfer Graduate School of Science Yeshiva University)