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Solutions, bifurcations and chaos of the nonlinear Schrödinger equation with weak damping*

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Using the wave packet theory, we obtain all the solutions of the weakly damped nonlinear Schrödinger equation. These solutions are the static solution, and solutions of planar wave, solitary wave, shock wave and elliptic function wave and chaos. The bifurcation phenomenon exists in both steady and non-steady solutions. The chaotic and periodic motions can coexist in a certain parametric space region.

Keywords: nonlinear Schrödinger equation, chaos, bifurcation
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1. Introduction

The nonlinear Schrödinger equation (NLSE) has very important applications in many scientific and technological fields, such as nonlinear optics, electromagnetic theory, superconductivity and superfluids, chemical and biological dynamics, lasers and plasmas.[1−6] Scientists have made many studies on this equation to uncover its properties.[1−11] The research results have shown that the NLSE possesses the characteristics of planar wave, solitary wave, shock wave, chaos and periodic motion.

In this paper, in terms of the wave packet theory, various solutions of the weakly damped NLSE are obtained, and its bifurcation property is revealed. The threshold of chaos and the coexistence of chaos and periodic motions are also obtained according to the Melnikov criterion.

2. Wave packet transformation for the weakly damped NLSE

We consider the weakly damped NLSE under the action of a small periodic perturbation

$$\mathrm{i}u_t + \alpha u_{xx} + \beta|u|^2u - \mathrm{i}\gamma u = h(x,t), \qquad (1)$$

where $\mathrm{i}=\sqrt{-1}$. u_t and u_{xx} denote the partial derivatives of function u with respective to time t and coordinate x, respectively; α, β and γ are all real constants; and $\mathrm{i}u_t$, αu_{xx}, $\beta|u|^2u$ and $\mathrm{i}\gamma u$ are the nonsteady, dispersive, nonlinear and complex damping terms. $h(x,t)$ is the small perturbation acting on the system, and it is a periodic function of time t and coordinate x. In the case of interactions between protein molecules and a weak laser field,[7] we have $h(x,t) = \eta \mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}\partial u/\partial_x$, and for a nonpropagating soliton excited by a vertical oscillation,[16] we have $h(x,t) = \eta u\mathrm{e}^{\mathrm{i}\omega_0 t}$. In this paper, to simplify the calculation, we set $h(x,t) = \eta u\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}$, where η is a small real constant.

We introduce the following wave envelope transformation

$$u(x,t) = A(\xi)\mathrm{e}^{\mathrm{i}(kx-\omega t+\varphi(\xi))}, \qquad (2)$$

where k is the wavenumber and ω denotes the angular frequency. $A(\xi)$ and $\varphi(\xi)$ are real functions of variable ξ , and

$$\xi = \varepsilon(x - c_gt). \qquad (3)$$

Here c_g is the group velocity of a wave packet, and $\varepsilon \ll 1$ is a small parameter. According to Eq.(3), we have

$$\frac{\partial}{\partial t} = -\varepsilon c_g\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial \xi}. \qquad (4)$$

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Inserting Eqs.(2) and (4) into Eq.(1) yields

$$\begin{aligned}&(\omega-\alpha k^2+\beta A^2)A+\varepsilon(c_g-2\alpha k)A\varphi_\xi\\&+\varepsilon^2\alpha(A_{\xi\xi}-A\varphi_\xi^2)\\=&\eta A\cos(\omega_0'\xi-\theta),\end{aligned}\tag{5a}$$

$$\begin{aligned}&-\gamma A-\varepsilon(c_g-2\alpha k)A_\xi+\varepsilon^2\alpha(2A_\xi\varphi_\xi+A\varphi_{\xi\xi})\\=&\eta A\sin(\omega_0'\xi-\theta),\end{aligned}\tag{5b}$$

where $\omega_0'=\omega_0/c_g,\theta=(k_0-\omega_0')x$. Since the damping $\mathrm{i}\gamma u$ and $\eta A\mathrm{e}^{\mathrm{i}(k_0x-\omega_0t)}$ are very small, we can rescale $\gamma A\to\varepsilon^3\gamma A,\eta A\to\varepsilon^3\eta A$. Thus Eqs.(5) become

$$\begin{aligned}&(\omega-\alpha k^2+\beta A^2)A+\varepsilon(c_g-2\alpha k)A\varphi_\xi\\&+\varepsilon^2\alpha(A_{\xi\xi}-A\varphi_\xi^2)\\=&\varepsilon^3\eta A\cos(\omega_0'\xi-\theta),\end{aligned}\tag{5a'}$$

$$\begin{aligned}&-\varepsilon(c_g-2\alpha k)A_\xi+\varepsilon^2\alpha(2A_\xi\varphi_\xi+A\varphi_{\xi\xi})-\varepsilon^3\gamma A\\=&\varepsilon^3\eta A\sin(\omega_0'\xi-\theta).\end{aligned}\tag{5b'}$$

3. Planar wave solutions and bifurcations

3.1. Zeroth-order solution of NLSE (1)

The zeroth-order form of Eqs.(5) can be obtained by setting $\varepsilon=0$,

$$(\omega-\alpha k^2+\beta A^2)A=0$$

or

$$[(\omega-\alpha k^2)/\beta+A^2]A=0.\tag{6}$$

The solutions of this equation read

$$A=0,\tag{7a}$$

$$A=\pm\sqrt{\lambda},\lambda=(\alpha k^2-\omega)/\beta,\lambda\geq0.\tag{7b}$$

Equations (7) indicate that, in the zeroth-order of NLSE (1), there is one fixed point when parameter $\lambda<0$, and there are three fixed points when parameter$\lambda>0$. The pitchfork bifurcation occurs when $\lambda=0$. In terms of the theory of the differential dynamical system,[12] we can determine the stability of the zeroth-order approximate steady solutions. Noting that Eq.(7) is the equilibrium point equation of the differential dynamical system

$$\dot{A}=f(A)=(\lambda-A^2)A,$$

where $\dot{A}=\dfrac{\mathrm{d}A}{\mathrm{d}\xi}$, then the Jacobian of this equation at the equilibrium point reads

$$Df=\lambda-3A^2=\begin{cases}\lambda, & \text{for } A=0,\\-2\lambda, & \text{for } A=\pm\sqrt{\lambda}.\end{cases}$$

Therefore, at the equilibrium point $A=0,Df(0)<0$, the eigenvalue of $Df(0)$ is negative as $\lambda<0$, and $Df(0)>0$, hence the eigenvalue of $Df(0)$ is positive as $\lambda>0$; and at the equilibrium points $A=\pm\sqrt{\lambda}$, when $\lambda<0$, $Df(\pm\sqrt{\lambda})>0$, hence the eigenvalue of $Df(\pm\sqrt{\lambda})$ is positive and when $\lambda>0$, $Df(\pm\sqrt{\lambda})<0$. Thus, the eigenvalue of $Df(\pm\sqrt{\lambda})$ is negative. So, according to the stability theorem of the equilibrium point,[12] the equilibrium point $A=0$ is stable for $\lambda<0$ and unstable for $\lambda>0$, whereas the equilibrium point $A=\pm\sqrt{\lambda}$ cannot exist for $\lambda<0$ and is stable for $\lambda>0$. The bifurcation diagram is plotted in Fig.1.

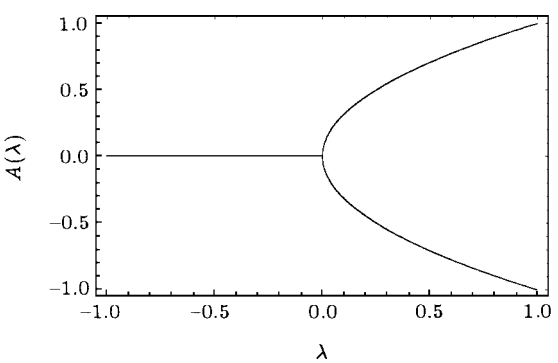


Fig.1. Bifurcation diagram.

The combination of representation (2) with solution (7b) tells us that the motion of NLSE (1) is a planar wave in the zeroth-order approximation and the planar wave only exists in the parameter space $\lambda>0$.

3.2. The first-order solution of NLSE (1)

Neglecting the terms with ε^2 or ε^3 in Eqs.(5) yields

$$\begin{aligned}&A_\xi=0,\\&\varepsilon(c-2\alpha k)\varphi_\xi=-(\omega-\alpha k^2+\beta A^2).\end{aligned}\tag{8}$$

Solving Eqs.(8), one has

$$A(\xi)=A_0=\text{const.},$$

$$\begin{aligned}\varphi(\xi)=&-\frac{\omega-\alpha k^2+\beta A^2}{\varepsilon(c_g-2\alpha k)}\xi\\=&-\frac{\omega-\alpha k^2+\beta A^2}{(c_g-2\alpha k)}(x-c_gt).\end{aligned}\tag{9}$$

Substituting Eqs.(9) into Eq.(2), we obtain the first-order approximate solution of NLSE (1) which is a plane wave.

3.3. The second-order solution of NLSE (1)

Adding Eq.(5b') to Eq.(5a') yields

$$\begin{aligned}&(\omega-\alpha k^2+\beta A^2)A+\varepsilon(c_g-2\alpha k)(A\varphi_\xi-A_\xi)\\&+\varepsilon^2\alpha(A_{\xi\xi}-A\varphi_\xi^2+2A_\xi\varphi_\xi+A\varphi_{\xi\xi})-\varepsilon^3\gamma A\\=&\varepsilon^3\eta' A\cos(\omega_0'\xi+\theta'),\end{aligned}\tag{10a}$$

$$\begin{aligned}&-\varepsilon(c_g-2\alpha k)A_\xi+\varepsilon^2\alpha(2A_\xi\varphi_\xi+A\varphi_{\xi\xi})-\varepsilon^3\gamma A\\=&\varepsilon^3\eta A\sin(\omega_0'\xi-\theta),\end{aligned}\tag{10b}$$

where $\eta'=\sqrt{2}\eta$, $\theta'=\theta+3\pi/4$. For simplification, we usually set $c_g-2\alpha k=0$, and $\varphi_\xi=0$.[13] Then Eqs.(10) read

$$\begin{aligned}&(\omega-\alpha k^2+\beta A^2)A+\varepsilon^2\alpha(A_{\xi\xi}+A\varphi_{\xi\xi})-\varepsilon^3\gamma A\\=&\varepsilon^3\eta' A\cos(\omega_0'\xi+\theta'),\end{aligned}\tag{11a}$$

$$\varepsilon^2\alpha A\varphi_{\xi\xi}-\varepsilon^3\gamma A=\varepsilon^3\eta A\sin(\omega_0'\xi-\theta).\tag{11b}$$

Neglecting the terms with ε^3 in Eq.(11b), we have

$$\varphi_{\xi\xi}=0.\tag{12}$$

Inserting this into Eq.(11a), we have

$$\begin{aligned}&(\omega-\alpha k^2+\beta A^2)A+\varepsilon^2\alpha A_{\xi\xi}-\varepsilon^3\gamma A\\=&\varepsilon^3\eta' A\cos(\omega_0'\xi+\theta').\end{aligned}\tag{13}$$

This is a Duffing equation with a small perturbation. Its equivalent form is

$$\begin{aligned}y_\xi&=z,\\z_\xi&=ay-by^3+\varepsilon[cy+\eta_0 y\cos(\omega_0'\xi+\theta')],\end{aligned}\tag{14}$$

where $a=\dfrac{\alpha k^2-\omega}{\varepsilon^2\alpha}, b=\dfrac{\beta}{\varepsilon^2\alpha}, c=\dfrac{\gamma}{\alpha}, \eta_0=\dfrac{\eta'}{\alpha}$, and $y(\xi)=A(\xi), z(\xi)=A_\xi(\xi)$. The unperturbed equations (14), namely, the second-order form of Eq.(13), can be expressed as

$$\begin{aligned}y_\xi&=z,\\z_\xi&=ay-by^3.\end{aligned}\tag{15}$$

This is a Hamilton system and its Hamiltonian is $h=\dfrac{1}{2}z^2-\dfrac{a}{2}y^2+\dfrac{b}{4}y^4$; its value is determined by the initial condition. The steady solutions of Eqs.(15) read

$$\left.\begin{matrix}y=0\\z=0\end{matrix}\right\},\ \text{and}\quad \left.\begin{matrix}y=\pm\sqrt{\lambda}, \lambda=a/b=(\alpha k^2-\omega)/\beta\\z=0.\end{matrix}\right\}\tag{16}$$

That is, the second-order solutions of Eq.(5) are $A=0$, and $A=\pm\sqrt{\lambda}$. These are the zero solution and planar wave solution of NLSE (1) respectively.

In terms of vector field bifurcation theory,[14] the solutions (16) show that there is only one equilibrium point $(0,0)$ in the phase plane (y,z) when $\lambda<0$. There are three equilibrium points $(0,0)$ and $(\pm\sqrt{\lambda},0)$ when $\lambda>0$; thus at the point $\lambda=0$, bifurcation occurs. The stability of these equilibrium points will be studied in another paper.

4. Solitary wave and shock wave

It is convenient for discussion to assume $\beta>0$. Thus, when $h=0$, the non-steady solutions of unperturbed equations (15) are the well-known homoclinic orbit for $\alpha<0$[13,18]

$$\begin{aligned}A=y_0&=\pm\sqrt{2\frac{|\alpha k^2-\omega|}{\beta}}\operatorname{sech}\sqrt{\frac{\alpha k^2-\omega}{\varepsilon^2\alpha}}\xi,\\A_\xi=z_0&=\pm\frac{|\alpha k^2-\omega|}{\varepsilon}\sqrt{-\frac{2}{\alpha\beta}}\\&\quad\times\operatorname{sech}\sqrt{\frac{\alpha k^2-\omega}{\varepsilon^2\alpha}}\xi\cdot\tanh\sqrt{\frac{\alpha k^2-\omega}{\varepsilon^2\alpha}}\xi,\end{aligned}\tag{17}$$

and in the homoclinic loop these are full of subharmonic orbits

$$\begin{aligned}y_m(\xi)&=\pm\sqrt{\frac{2(\omega-\alpha k^2)}{(1+m^2)\beta}}m\\&\quad\times\operatorname{sn}\left(\sqrt{\frac{\alpha k^2-\omega}{(1+m^2)\varepsilon^2\alpha}}\xi\right)\pm\sqrt{\frac{\omega-\alpha k^2}{\beta}},\\z_m(\xi)&=\pm\frac{(\alpha k^2-\omega)}{(1+m^2)\varepsilon\alpha}m\sqrt{\frac{-2\alpha}{\beta}}\\&\quad\times\operatorname{cn}\left(\sqrt{\frac{\alpha k^2-\omega}{(1+m^2)\varepsilon^2\alpha}}\xi\right)\\&\quad\times\operatorname{dn}\left(\sqrt{\frac{\alpha k^2-\omega}{(1+m^2)\varepsilon^2\alpha}}\xi\right),\end{aligned}\tag{18}$$

with period $T=4K(m)\sqrt{\dfrac{(1+m^2)\varepsilon^2\alpha}{\alpha k^2-\omega}}$. The subharmonic motion corresponds to $h=\dfrac{m^2a^2}{(1+m^2)^2b}$. Here, cn ( ), dn ( ) and sn ( ) are Jacobi elliptic functions, $K(\ )$ is the completely elliptic integral of the first kind, and m is the modulus. The first equation of Eqs.(17) represents a solitary wave which is

illustrated in Fig.2. Equations (18) denote the elliptic function waves. Similarly, we can easily obtain the shock wave solution (Fig.3) of NLSE (1) for $\alpha > 0$.

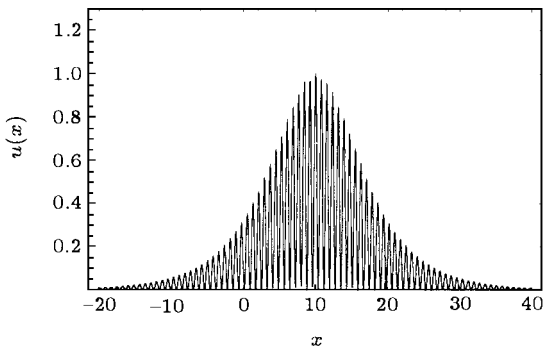


Fig.2. Solitary wave diagram.

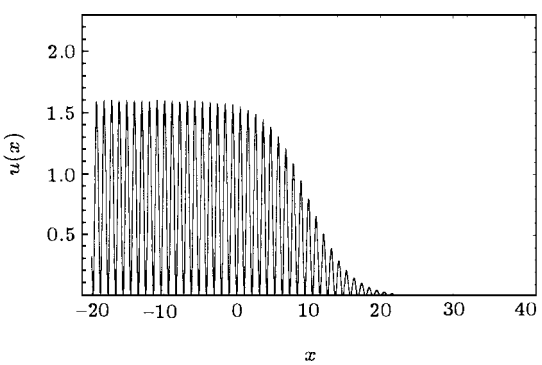


Fig.3. Shock wave diagram.

5. Chaos

We can prove that NLSE (1) possesses a chaotic behaviour with the Melnikov criterion.[14] For this, we consider the time evolution of Eqs.(14), which means that in Eqs.(14) x is fixed so that θ is a constant. Hence, the Melnikov function of the homoclinic orbit (17) can be calculated as follows

$$M_0(\xi_0) = I \cdot \eta_0 \sin \omega_0' \xi_0 + \gamma \cdot 0, \tag{19}$$

$$I = \pm \int_{-\infty}^{\infty} \frac{|\alpha k^2 - \omega|}{\varepsilon} \sqrt{-\frac{2}{\alpha\beta}} \operatorname{sech}^2 \sqrt{\frac{\alpha k^2 - \omega}{\varepsilon^2 \alpha}} \xi \times \tanh \sqrt{\frac{\alpha k^2 - \omega}{\varepsilon^2 \alpha}} \xi \cdot \sin \omega_0' \xi \cdot \mathrm{d}\xi. \tag{20}$$

It is obvious that $M_0(\xi_0)$ possesses a simple zero point. This indicates that the solitary wave exhibits a chaotic behaviour. The threshold $R_0(\omega_0')$ of chaos is

$$\eta_0/\gamma \geq 0 = R_0(\omega_0'). \tag{21}$$

According to this equation we can conclude that, if the damping is only related to u other than on u_ξ, then chaotic motion occurs in system (1) under a periodic perturbation, even though the perturbation is very small. It is consistent with recent discoveries that the motion of biology molecules will change from solitary wave into chaos under the action of a weak laser.[8,10,15]

If the angular frequency ω_0' of the periodic perturbation satisfies the condition

$$T = 4K(m)\sqrt{\frac{(1+m^2)\varepsilon^2\alpha}{\alpha k^2 - \omega}} = \frac{2\pi l}{n\omega_0'} \tag{22}$$

(l, n are relatively prime numbers), the periodic perturbation will give rise to the sub-harmonic resonance in system (14). After a complicated calculation, we obtain the threshold of sub-harmonic bifurcation

$$R_1(\omega_0') = \frac{\varepsilon}{2\pi\omega_0'(1+m^2)} \sqrt{\frac{\omega - \alpha k^2}{\beta}} \times \sinh\left(\omega_0' K'(m) \sqrt{\frac{(1+m^2)\varepsilon^2\alpha}{\alpha k^2 - \omega}}\right). \tag{23}$$

It is obvious that the sub-harmonic bifurcation threshold is larger than the chaos threshold of the homoclinic orbit, namely

$$R_1(\omega_0') > R_0(\omega_0'). \tag{24}$$

This implies that, in system (14), chaotic motion and periodic motion coexist in the region determined by the inequality (24). That is to say, in this region, both the chaotic motion and sub-harmonic periodic motion are likely to occur. Which of these occurs in a concrete situation is determined by the Hamiltonian of the system. When $h = 0$, the motion of the system is chaotic, and if $h = \dfrac{m^2 a^2}{(1+m^2)^2 b}$, the motion of the system is periodic.

Equation (14) is a derived system of NLSE (1). Equation (2) indicates that if $A(\xi)$ is chaotic, then the solution $u(x,t)$ of NLSE (1) is certainly chaotic. Since NLSE (1) is an infinite dimensional dynamical system, we cannot determine the threshold of chaos of NLSE(1) by means of the Melnikov criterion. The above discussion shows that if the Hamiltonian of system (15) is zero, then the motion of the system will come into chaos via a homoclinic bifurcation under a small periodic perturbation. Moreover, if omitting the weak damping and small periodic perturbation,

the NLSE (1) is a Hamiltonian system and its Hamiltonian corresponding to the homoclinic orbit of the derived system (14) is[17,19]

$$H = \int_{-\infty}^{+\infty} \{u_x^* u_x - (u^* u)^2\} \mathrm{d}x = 8\alpha^2 k^2 \sqrt{\alpha(\alpha k^2 - \omega)}. \quad (25)$$

This is the condition under which chaos can occur via a homoclinic bifurcation in NLSE (1). Therefore, if Eq.(25) is true, the solitary motion occurs in the system of NLSE (1), and if acted by a small periodic perturbation, the motion comes into chaos via homoclinic bifurcation.

6. Conclusion

By using the well-known method of wave packet transformation, we obtain the various solutions of the NLSE, such as planar wave solution, and solutions of solitary waves, shock waves and elliptic function waves. The bifurcation phenomenon of the steady solutions and the coexistence of the chaotic and periodic motions in the weakly damped NLSE are discovered. We also prove that if the damping of the NLSE only relies on the function u, then the solitary wave solution is unstable and any small periodic perturbation can make it come into chaotic motion.

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