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Symmetry groups and spiral wave solution of a wave propagation equation*

Zhang Quan-Ju(张全举)[a] and Qu Chang-Zheng(屈长征)[b]
[a] School of Science, Xidian University 710071, Xi'an, China
[b] Institute of Modern Physics, Northwest University, Xi'an 710069, China

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We study a third-order nonlinear evolution equation, which can be transformed to the modified KdV equation, using the Lie symmetry method. The Lie point symmetries and the one-dimensional optimal system of the symmetry algebras are determined. Those symmetries are some types of nonlocal symmetries or hidden symmetries of the modified KdV equation. The group-invariant solutions, particularly the travelling wave and spiral wave solutions, are discussed in detail, and a type of spiral wave solution which is smooth in the origin is obtained.

Keywords: Symmetry group, optimal system, wave propagation, spiral wave
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1. Introduction

This paper is devoted to the study of a third-order nonlinear evolution equation arising from surface diffusion

$$\frac{\partial N}{\partial t} = B\frac{\partial \kappa}{\partial s}, \tag{1}$$

where $B = \pm 1$, $\dfrac{\partial N}{\partial t}$ is the normal velocity of the curve, κ is the curvature of the curve, and s is the arclength. Equation (1) describes the wave propagation along a stretched rope[1] and governs long frontal waves in the quasi-geostrophic approximation.

In Cartesian coordinates or local graph, Eq.(1) can be written as

$$\frac{\partial u}{\partial t} = B\frac{\partial}{\partial x}\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}, \tag{2}$$

where B is given by $B = \mathrm{sgn}\left(\dfrac{\mathrm{d}x}{\mathrm{d}s}\right)$. Equation (2) is generally called the WKI equation and can be transformed to the modified KdV equation by transformations of dependent and independent variables.[2] In fact, introducing the nonlocal transformations

$$u_x = \tan\theta, \quad x = \int^s \cos\theta \mathrm{d}s, \quad \tau = t, \tag{3}$$

then $\theta(\tau, s)$ satisfies the modified KdV equation of the potential form

$$\theta_\tau + \theta_{sss} + \frac{1}{2}\theta_s^3 = 0,$$

and the curvature κ satisfies the modified KdV equation

$$\kappa_t + \kappa_{sss} + \frac{3}{2}\kappa^2\kappa_s = 0.$$

Equation (1) or Eq.(2) is a member $n = 1$ of the hierarchy

$$\frac{\partial N}{\partial t} = B\frac{\partial^n \kappa}{\partial s^n}. \tag{4}$$

For $n = 0$, it is the well-known Mullins equation[3] and this describes the curve-shortening problem (also referred to as the curve-shortening equation). Many interesting results about this equation have been obtained.[4–9] Another interesting case of the hierarchy (4) is for $n = 2$. In this case, Eq.(4) has been used by Mullins[3] with great success to model the formation of grooves at an interesting grain on exposing surface.

To date, there are almost no analytical or numerical results for Eq.(4) with $n = 2$. When $n = 1$, only loop solutions are known. The one-loop and two-loop solutions have been obtained by the inverse scattering method.[2,10] The N-loop soliton solution has been derived using the N-soliton solution of the modified KdV equation.[11] Since one can use the solutions of the modified KdV equation to obtain solutions of Eq.(2), a natural question is whether we can use the solutions of Eq.(2) to gain new solutions of the modified KdV equation.

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The goal of this paper is to investigate the symmetry groups and group-invariant solutions, particularly the travelling wave and spiral wave solutions of Eq.(1). It is known that the Lie symmetry group method and its various generalizations[12−17] are useful for finding symmetry reductions and exact solutions of nonlinear partial differential equations (PDEs). On the other hand, these are closely related to the direct method,[18−20] which has been used successfully to study symmetry reductions of nonlinear integrable or non-integrable equations. In this paper, we first determine the Lie algebra of all infinitesimal symmetries. According to the Lie theory, given any infinitesimal symmetry algebra V, or more precisely the subspace spanned by V, there is a corresponding V-invariant solution. All group-invariant solutions are classified in terms of the optimal system of symmetry algebras. Among group-invariant solutions, we pay more attention to the travelling wave and spiral wave solutions, since they are quite important in describing the behaviour of the water wave. For the KdV equation and its extensions, the travelling wave solutions are known.[21−23]

It has been found that the spiral wave frequently appears in biological, physiological and chemical contexts.[24] Several methods have been developed to study spiral wave solutions. But almost no analysis has considered the asymptotic spiral waves corresponding to the solutions that are smooth at the origin. In section 4, we obtain the spiral wave of Eq.(2) that is smooth at the origin.

2. Infinitesimal symmetries of equation (2)

As mentioned above, symmetry group methods are effective in tackling nonlinear PDEs. The most important and fundamental of these is the Lie classical method.[12] To apply the classical method to Eq.(2), we find the one-parameter Lie group of transformations in (x,t,u) given by

$$\begin{aligned}\tilde{x} =& x+\epsilon\xi(x,t,u)+O(\epsilon^2),\\ \tilde{t} =& t+\epsilon\tau(x,t,u)+O(\epsilon^2),\\ \tilde{u} =& u+\epsilon\phi(x,t,u)+O(\epsilon^2),\end{aligned}\tag{5}$$

where ϵ is the group parameter. The corresponding Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$V=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\phi(x,t,u)\partial_u.\tag{6}$$

According to the infinitesimal invariance criterion for symmetry, Eq.(2) is invariant under the transformations (5) if and only if

$$Pr^{(3)}V(\Delta[u])=0,\quad \text{on}\quad \Delta[u]=0,\tag{7}$$

where

$$\Delta[u]=\frac{\partial u}{\partial t}-B\frac{\partial}{\partial x}\frac{u_{xx}}{(1+u_x^2)^{3/2}},$$

and $Pr^{(3)}V$ is the third prolongation of V on the 3-jet space of (x,t,u),

$$Pr^{(3)}V=V+\phi^t\partial_{u_t}+\phi^x\partial_{u_x}+\phi^{xx}\partial_{u_{xx}}+\phi^{xxx}\partial_{u_{xxx}},$$

and from Eq.(7), we have

$$\begin{aligned}&\phi^t+B\phi^x[3u_x(1+u_x^2)^{-5/2}u_{xxx}\\ &+3(1+u_x^2)^{-5/2}u_{xx}^2-15(1+u_x^2)^{-7/2}u_x^2u_{xx}^2]\\ &+\phi^{xx}[u_x(1+u_x^2)^{-5/2}u_{xx}]\\ &-B\phi^{xxx}(1+u_x^2)^{-3/2}=0,\end{aligned}\tag{8}$$

whenever $\Delta[u]=0$, where ϕ^t, ϕ^x, ϕ^{xx} and ϕ^{xxx} are given explicitly in terms of ξ, τ and ϕ and their derivatives.[12] After eliminating u_t in Eq.(8) by using $\Delta[u]=0$, then the left-hand side of Eq.(8) becomes an algebraic polynomial in u_{xx}, u_{xxx} with coefficients being rational polynomial in u_x. Equating those coefficients to zero yields a linear PDE for ξ, τ and ϕ

$$\tau_x=\tau_u=0,\quad \phi_t=\xi_t=0,\quad \tau_t=3\xi_x,$$

$$\phi_x+\xi_u=0,\quad \xi_x-\phi_u=0,$$

which has a general solution

$$\begin{aligned}\tau =& c_1t+c_2,\\ \xi =& \frac{c_1}{3}x-c_4u+c_3,\\ \phi =& \frac{c_1}{3}u+c_4x+c_5,\end{aligned}$$

where c_i, $i=1,2,3,4,5$, are parameters. We denote the vector space of all infinitesimal symmetries of Eq.(2) by $s(\sigma)$. Then $s(\sigma)$ is spanned by

$$\begin{aligned}T =& \partial_t,\quad \text{(Translation in } t)\\ D =& x\partial_x+u\partial_u+3t\partial_t,\quad \text{(Dilatation)}\\ V_1 =& \partial_x,\quad \text{(Translation in } x)\\ V_2 =& \partial_u,\quad \text{(Translation in } u)\\ V_3 =& x\partial_u-u\partial_x,\quad \text{(Rotation from } x-\text{axis to } u-\text{axis})\end{aligned}$$

The structure of $s(\sigma)$ is shown in Table 1.

Table 1. The structure of Lie algebra $S(\sigma)$.

$S(\sigma)$	T	D	V_1	V_2	V_3
T	0	$3T$	0	0	0
D	$-3T$	0	$-V_1$	$-V_2$	0
V_1	0	V_1	0	0	V_2
V_2	0	V_2	0	0	$-V_1$
V_3	0	0	$-V_2$	V_1	0

We call the Lie algebra, with the structure given in Table 1, the pseudo-affine Lie algebra, and the corresponding Lie group $S(\sigma)$ the pseudo-affine Lie group. This plays an important role in surface diffusion and evolution. Due to transformations (3), we have shown that the modified KdV equation admits the pseudo-affine group as the nonlocal symmetry group or hidden symmetry group.

Since the linear combinations of any one-dimensional subalgebra is also a subalgebra, generally, it is impossible to write down all the reduced equations, let alone to solve them. So it is necessary to classify them. We say that two subalgebras of η_1 and η_2 are equivalent if $g \in S(\sigma)$ exists such that $Ad(g)(\eta_1) = \eta_2$, where $Ad(g)$ is the adjoint representation of g on $S(\sigma)$. An optimal system of one-dimensional subalgebras is a collection of inequivalent one-dimensional subalgebras $\{\eta_\alpha\}_{\alpha\in A}$. Once the optimal system has been obtained, all the group-invariant solutions can be obtained as the images of the η_α-invariant solution under adjoint representation. We find that an optimal system for the Lie algebra $S(\sigma)$ is generated by

$$W_1 = V_1, \quad W_2 = T, \quad W_3 = V_2 + T, \quad W_4 = V_3,$$

$$W_5 = V_3 + T, \quad W_6 = V_3 - T,$$

$$W_7 = D, \quad W_8 = D + \alpha V_3.$$

Note that W_8 contains infinitely many elements.

Now we list the reduced equations corresponding to vector fields of the optimal system in Table 2, where d_1 is an integration constant and $\lambda = 2yv_y$.

Table 2. The reduced equations of Eq.(2).

W_i	Invariants (y, v)	Reduced equations
W_1	(t, u)	$v_y = 0$
W_2	(x, u)	$\left(\frac{v_{yy}}{(1+v_y^2)^{3/2}}\right)_y = d_1$
W_3	$(x, u-t)$	$\left(\frac{v_{yy}}{(1+v_y^2)^{3/2}}\right)_y = y + d_1$
W_4	(t, x^2+u^2)	$v_y = 0$
W_5	$\left(x^2+u^2, \arctan\frac{u}{x} - t\right)$	$\frac{\mathrm{d}\lambda}{\mathrm{d}y} = \frac{1}{2}(1+\lambda^2)\left[y^{-1/2}\left(\frac{1}{2}y + d_1\right)(1+\lambda^2)^{1/2} - \frac{\lambda}{y}\right]$
W_6	$\left(x^2+u^2, \arctan\frac{u}{x} + t\right)$	$\frac{\mathrm{d}\lambda}{\mathrm{d}y} = \frac{1}{2}(1+\lambda^2)\left[y^{-1/2}\left(d_1 - \frac{1}{2}y\right)(1+\lambda^2)^{1/2} - \frac{\lambda}{y}\right]$
W_7	$\left(xt^{-1/3}, \frac{u}{x}\right)$	$\left(\frac{v_{yy}}{(1+v_y^2)^{3/2}}\right)_y = \frac{1}{3}(v - yv_y)$
W_8	$\left(t^{-1/2}(x^2+u^2), \arctan\frac{u}{x} - \frac{\alpha}{3}\ln t\right)$	$\left(\frac{\lambda(1+\lambda^2)+2y\lambda_y}{y^{1/2}(1+\lambda^2)^{3/2}}\right)_y = \frac{\alpha-\lambda}{6}$

3. Stationary and travelling wave solutions

The group-invariant solution corresponding to $W_2(=\partial_t)$ is the stationary solution, that is, the evolution curve which does not depend on time t. The curve satisfies a second-order ordinary differential equation (ODE)

$$\frac{u_{xx}}{(1+u_x^2)^{3/2}} = d_1, \tag{9}$$

which implies that the curvature of the curve is constant, and the solution is a straight line or a circle.

The group-invariant solution with respect to W_3

is the travelling wave solution. It takes the form

$$u = t + v(x), \tag{10}$$

where $v(x)$ satisfies the ODE

$$\frac{v_{xx}}{(1+v_x^2)^{3/2}} = x + d_1. \tag{11}$$

Integration of Eq.(11) gives

$$v_x = \frac{z^2 + 2d_2}{\sqrt{4-(z^2+2d_2)^2}},\ z = x + d_1, \tag{12}$$

where d_2 is an integration constant. To proceed, we need to distinguish four cases:

(1) $d_2 = -1$. The solution $v(x)$ is given by

$$v(x) = -\sqrt{4-z^2} - \ln\frac{z}{2+\sqrt{4-z^2}} + d_3. \tag{13}$$

Here and hereafter d_i, $i = 1, 2, ...$, are arbitrary constants. This solution is well defined in the region $-2 \le z \le 2$ and, when introducing a new parameter σ, can be written as

$$z = 2\ \mathrm{sech}\ \sigma,$$

$$v = \sigma - 2\tanh\sigma + d_3,$$

which gives the one-loop soliton of Eq.(2).

(2) $-1 < d_2 < 1$. The solution of Eq.(11) is given by

$$\begin{aligned} v(x) =& 2E\left(\arcsin\left(\sqrt{\frac{4}{z^2+2+2d_2}}\frac{z}{\sqrt{1-d_2}}\right), \sqrt{\frac{1-d_2}{2}}\right) \\ &+ \frac{\sqrt{2}}{2}F\left(\arcsin\left(\sqrt{\frac{4}{z^2+2+2d_2}}\frac{z}{\sqrt{1-d_2}}\right), \right. \\ &\left.\sqrt{\frac{1-d_2}{2}}\right) + d_3, \end{aligned} \tag{14}$$

where

$$E(\phi, k) = \int_0^\phi \sqrt{1-k^2\sin^2\alpha}\mathrm{d}\alpha,$$

and

$$F(\phi, k) = \int_0^\phi \frac{1}{\sqrt{1-k^2\sin^2\alpha}}\mathrm{d}\alpha$$

are the elliptic integrals, and k is the modulus of the elliptic integral.

(3) $d_2 < -1$.

In the interval $0 < b < z \le a$, where $b = \sqrt{-2(1+d_2)}$, $a = \sqrt{2(1-d_2)}$, the corresponding solution of Eq.(11) is given by

$$\begin{aligned} v(x) =& aE(\chi, q) + \frac{4d_2+3}{2a}F(\chi, q) \\ &- \frac{1}{z}\sqrt{(a^2-z^2)(z^2-b^2)} + d_3, \end{aligned} \tag{15}$$

where

$$\chi = \arcsin\frac{a}{z}\sqrt{\frac{z^2-b^2}{4}},\quad q = \sqrt{\frac{2}{1-d_2}}.$$

In the interval $b \le z < a$, a solution of Eq.(11) is given by

$$v(x) = -aE(\nu, q) - \frac{4d_2+5}{2a}F(\nu, q) + d_3, \tag{16}$$

with $\nu = \arcsin\sqrt{(a^2-z^2)/4}$.

(4) $d_2 > 1$. In this case, no solution exists.

Remark In general, the travelling wave is of the form $u = f(x-ct)$. Since Eq.(2) is invariant under the rotation group of x and u, we may take the travelling wave solution in the form (10).

4. Spiral wave solution of Eq.(2) (B=1)

The group-invariant solution corresponding to $W_5 = \partial_t - u\partial_x + x\partial_u$ is the spiral wave solution, which can be described in the (x, u)-plane by polar coordinates

$$x = r\cos\theta(y,t),\quad u = r\sin\theta(y,t),\quad y = r^2,$$

where $\theta = \theta(y,t)$ satisfies

$$\begin{aligned} \theta &= \phi(y) + t, \\ \frac{\mathrm{d}\phi}{\mathrm{d}y} &= \frac{\lambda}{2y}, \end{aligned} \tag{17}$$

and

$$\frac{\mathrm{d}\lambda}{\mathrm{d}y} = \frac{1}{2}(1+y^2)\left[\left(\frac{y}{2}+d_1\right)y^{-1/2}(1+\lambda^2)^{1/2} - \frac{\lambda}{y}\right], \tag{18}$$

where d_1 is a constant. In what follows we choose $d_1 = 0$, without loss of generality. Substituting Eq.(17) into Eq.(18) yields

$$[\arctan\lambda(y)]' = \frac{1}{4}y^{1/2}(1+\lambda^2)^{1/2} - \phi'(y).$$

Hence

$$\phi(y) = -\arctan\lambda(y) + \frac{1}{4}\int_0^y y^{1/2}(1+\lambda^2)^{1/2}\mathrm{d}y.$$

A simple analysis of Eq.(18) is presented as follows. Setting the right-hand side to zero, we have

$$\lambda = \frac{1}{2}y^{3/2}(1+\lambda^2)^{1/2}. \tag{19}$$

Equation (19) defines a curve $\Gamma = (y, \Lambda(y))$ in $(0, 2^{\frac{2}{3}}) \times R^+$, where $\Lambda(y)$ is strictly increasing and satisfies

$$\begin{aligned} &\lim_{y\to 0}\Lambda(y) = 0,\quad \lim_{y\to 0}\frac{2\Lambda(y)}{y^{3/2}} = 1, \\ &\lim_{y\to 2^{2/3}}\Lambda(y) = +\infty. \end{aligned} \tag{20}$$

The curve Γ and the straight line $y = 2^{2/3}$ divide $(0,\infty) \times \boldsymbol{R}$ into three regions: $R_1 = \left\{(y,\lambda(y)),\ y \in (0,2^{2/3}),\ \frac{\lambda}{(1+\lambda^2)^{2/3}} > \frac{1}{2}y^{3/2}\right\}$, $R_2 = \left\{(y,\lambda(y)),\ y \in (0,2^{2/3}),\ \frac{\lambda}{(1+\lambda^2)^{2/3}} < \frac{1}{2}y^{3/2}\right\}$ and $R_3 = \{(y,\lambda(y)),\ y \in (2^{2/3},+\infty),\ \lambda \in (-\infty,+\infty)\}$.

Any integral curve of Eq.(18) starting from region R_1 strictly decreases as y increases until it hits Λ, then it starts to increase and tend to Λ asymptotically, never crossing Λ again. The second case is that any integration curve starting from R_2 strictly increases and tends to Λ from below. The third case is that any integration curve starting from R_2 strictly increases and crosses $y = 2^{2/3}$ and, if satisfying $\Lambda(y)$, will tend to infinity as $y \to \infty$. But we claim that the third case cannot occur, which can be seen from Eq.(18).

Ω^+ denotes the union of all integral curves starting in the upper region R_1, and Ω^- denotes the union of all integral curves starting in R_2 and hitting $\Lambda(y)$. We set

$$\lambda^*(y) = \inf\{\lambda; (y,\lambda) \in \Omega^+\},$$

$$\lambda_*(y) = \sup\{\lambda; (y,\lambda) \in \Omega^-\}.$$

By the continuity and compactness of the solutions of Eq.(18), both λ^* and λ_* are really seen to be solutions of Eq.(18) satisfying $\lambda_*(0) = \lambda^*(0) = 0$. Moreover

$$0 \le \lambda_* \le \lambda^* \le \Lambda$$

in $[0, 2^{2/3}]$. We rewrite Eq.(18) as

$$(\sqrt{y}\lambda)_y = \frac{1}{4}y(1+\lambda^2)^{3/2} - \frac{\lambda^3(y)}{2\sqrt{2}}.$$

Then for a solution satisfying $\lambda(0) = 0$, we have

$$\sqrt{y}\lambda(y) = \frac{1}{4}\int_0^y (1+\lambda^2(s))^{3/2} s\mathrm{d}s - \int_0^y \frac{\lambda^3(s)}{2\sqrt{s}}\mathrm{d}s.$$

By Eq.(20) and the definition of Λ,

$$0 \le \sqrt{y}\lambda(y) \le \sqrt{y}\Lambda(y) \le Cy^2,\ C = \text{Const.} > 0.$$

Therefore

$$\sqrt{y} = \lambda = \frac{C}{8}y^2(1+O(1)).$$

We call the unique maximal solution starting at R_1 (resp. R_2) by λ^+ (resp. λ^-) a type I (type II) solution. Note that for all solutions, Eq.(19) holds for all λ_0. It can be seen that, for any λ^+ and λ^-, $0 < \delta < 2^{2/3}$ exists such that

$$\lim_{y\downarrow\delta} \lambda^+(y) = +\infty,$$

$$\lim_{y\uparrow\delta} \lambda^-(y) = -\infty.$$

Furthermore, unique type I and type II solutions exist which blow up at δ. The graphs of type I, type II solutions and λ_0 form a foliation of $(0,+\infty) \times R$.

The solution curve (17) depends on two parameters since ϕ is essentially of second order. One parameter is the choice of λ and the other is the integration constant from integrating Eq.(17). The latter accounts for a rotation of the solution curve of Eq.(17) in the (x,u)-plane, and hence is not essential. We always assume that it has been chosen so that the solution curve starts at the positive x-axis or the origin. In fact, we have: (a) When λ is of type I (resp. λ_0) defined in the maximal interval $(\delta, 2^{2/3})$ (resp. $[0,\infty)$), then θ increases from 0 to ∞ as y increases from δ to ∞. (b) When λ is of type II, defined in the maximal interval $(\delta, 2^{2/3})$, then $\delta' > \delta$ exists such that θ decreases for (δ,δ') and then increases in $(\delta', 2^{2/3})$. Therefore, it is possible to match type I and type II solutions together to obtain a smooth complete solution curve.

The other two interesting group-invariant solutions which correspond to W_7 and W_8, respectively, are the self-similar solution and contracting spiral wave solution ($\alpha < 1$) (resp. expanding spiral wave solution ($\alpha > 1$)). After these two solutions, in terms of transformation (3), we can obtain the corresponding exact solutions of the modified KdV equation. We do not consider these here.

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