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# Time dependence of the average charge and current in a dissipative mesoscopic circuit*

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Taking into consideration the interactions between electrons and phonons, we have studied the temporal evolution of the average charge and current in a dissipative mesoscopic *RLC* circuit. Our results show that a mesoscopic *RLC* circuit can be treated as an interactive system between an electromagnetic harmonic oscillator and many lattice harmonic oscillators; this is called the bathing of the harmonic oscillators. The results also show that the quantum equation of motion of the linear mesoscopic *RLC* circuit is identical in form to its classical equation of motion, the only difference between them being their respective meanings. In order to thoroughly study the quantum properties of a dissipative mesoscopic circuit, we have to consider not only the electromagnetic energy of the circuit, but also the crystal lattice vibration energy and the interactive energy between electrons and phonons.

**Keywords:** dissipative mesoscopic *RLC* circuit, interaction between electrons and phonons
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## 1. Introduction

When the dimension of a circuit or an electronic element is comparable with the phase coherent length of an electron, the quantum effect of the circuit should be considered. Recently, much research has been carried out on the quantum effect of the non-dissipative mesoscopic circuit.[1−7] Some researchers have studied the influence of a resistance, whose macroscopic parameter is $R$, on the quantum effect in the dissipative mesoscopic circuit.[8−11] But most have only considered the effect of the electromagnetic energy and neglected the microscopic mutual action between the system and environment. Feynman and Vernon (see [12]) have proposed a theory about the mutual action between the quantum system and environment. In their theory, they have used the concept of bathing of oscillators to express the environment. According to this theory, other researchers[13−15] have considered the environment as an aggregate of a set of harmonic oscillators $Q_\alpha$ whose mass and frequency are $m_\alpha$ and $\omega_\alpha$, respectively. As is known, the resistance of a circuit is the result of collisions between the crystal lattice and electrons, and the crystal lattice oscillation is equivalent to the movement of phonons. So the mesoscopic *RLC* circuit can be equivalent to an interactive system between electromagnetic harmonic oscillators and the environment bath of the lattice oscillators. In this paper, we study the Hamiltonian of the system and the time dependence of the mean charge and current. Our results indicate that the quantum fluctuations of the charge and current are not related to the initial state of the system in its stable state.

## 2. Quantum scheme

The classical kinetic equation of an *RLC* circuit is

$$L\ddot{q} + R\dot{q} + C^{-1}q = \varepsilon(t), \tag{1}$$

where $L$, $C$ and $R$ are the inductance, capacitance and resistance, respectively, $\varepsilon(t)$ is the external voltage source, and $q(t)$ is the charge that is regarded as a coordinate of the system. Equation (1) shows that the movement of the system is equivalent to that of a forced damped harmonic oscillator. Obviously, the dissipative term of the equation is concerned with the first derivative of the coordinate, and its coefficient is the resistance of the system. The generalized current can be defined as $p(t) = L\dot{q}(t)$. If the lattice vibration energy is neglected, Refs.[8] and [9] provide the expression for quantum fluctuation of charge and current in the ground state, as follows

$$\overline{(\Delta q)^2} = \frac{\hbar}{2L\Omega_0}, \tag{2a}$$

$$\overline{(\Delta p)^2} = \frac{\hbar L\omega_0^2}{2\Omega_0}, \tag{2b}$$

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where $\Omega_0 = \sqrt{\omega_0^2 - \beta^2}$, $\omega_0^2 = (LC)^{-1}$, $\beta = R/2L$.

Evidently, the non-dissipative mesoscopic $LC$ circuit is equivalent to an electromagnetic harmonic oscillator. According to solid-state physics, resistance is the result of electron scattering by the crystal lattice, and the lattice oscillation can be equivalent to the movements of a large number of phonons under the harmonic approximation. Therefore the dissipative mesoscopic $RLC$ circuit can be regarded as a harmonic oscillator coupled with many phonons, which is called the bathing of harmonic oscillators.

The total energy of the system includes the following three types of energy: the electromagnetic energy of the circuit; the crystal lattice energy; the interacting energy. The electromagnetic energy of the circuit is expressed as

$$E_0 = \frac{1}{2}L\dot{q}^2 + \frac{1}{2C}q^2 - q\varepsilon(t).$$

Obviously, $q$ depends on the resistance $R$ of the $RLC$ circuit. In this paper resistance is treated as a set of normal mode oscillators, i.e. a group of phonons. We let $Q_\alpha$, $m_\alpha$ and $\omega_\alpha$ represent the coordinates, mass and frequency of the phonons respectively, and $\alpha$ is the mode number. Then the energy of a crystal lattice ($m_\alpha$=1) is

$$E_{\rm p} = \frac{1}{2}\sum_\alpha (\dot{Q}_\alpha^2 + \omega_\alpha^2 Q_\alpha^2).$$

The mode of interactions between electrons and phonons can be treated as the coupling of momentum and coordinate. Then the interactive energy is expressed as[12]

$$E_{\rm e-p} = -q\sum_\alpha c_\alpha \dot{Q}_\alpha,$$

where $c_\alpha$ is the coupling coefficient.

If there is no external power supply in the circuit, the classical Lagrangian of this system is

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}L\dot{q}^2 - \frac{1}{2C}q^2 - q\sum_\alpha c_\alpha \dot{Q}_\alpha \\ &+ \frac{1}{2}\sum_\alpha [\dot{Q}^2 - \omega_\alpha^2 Q_\alpha^2]. \end{aligned} \tag{3}$$

From this Lagrangian, we can obtain the equations of motion by $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{X}}\right) - \dfrac{\partial \mathcal{L}}{\partial X} = 0$,

$$L\ddot{q} + L\omega_0^2 q = -\sum_\alpha c_\alpha \dot{Q}_\alpha, \tag{4a}$$

$$\ddot{Q}_\alpha + \omega_\alpha^2 Q_\alpha = c_\alpha \dot{q} \qquad \alpha = 1, 2, 3, \cdots \tag{4b}$$

where $\omega_0^2 = (LC)^{-1}$. The conjugate momentum $p$ and $P_\alpha$ can be obtained from

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = L\dot{q}, \tag{5a}$$

$$P_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{Q}_\alpha} = \dot{Q}_\alpha - qc_\alpha. \tag{5b}$$

Then, the Hamiltonian of the system is

$$\begin{aligned} H =& p\dot{q} + \sum_\alpha P_\alpha \dot{Q}_\alpha - \mathcal{L} \\ =& \frac{1}{2L}p^2 + \frac{1}{2C}q^2 + \frac{1}{2}\sum_\alpha (P_\alpha^2 + \omega_\alpha^2 Q_\alpha^2) \\ &+ q\sum_\alpha c_\alpha P_\alpha + \frac{1}{2}q^2\sum_\alpha c_\alpha^2. \end{aligned} \tag{6}$$

According to the canonical quantization principle, we can quantize the system by operators $(q, p)$ and $(Q_\alpha, P_\alpha)$, which satisfy the following commutation relations, respectively, $[q, p] = \mathrm{i}\hbar$, $[Q_\alpha, P_\beta] = \mathrm{i}\hbar\delta_{\alpha\beta}$. The Heisenberg equation of a quantum-mechanical operator $X$ is

$$\dot{X} = -\frac{\mathrm{i}}{\hbar}[X, H], \tag{7}$$

where $X \in (q, p, Q_\alpha, P_\alpha)$. Actually, we may also derive Eqs.(4a) and (4b) from Eq.(7). This shows that the quantum equation of motion of the linear mesoscopic $RLC$ circuit is identical in form to its classical equation of motion, where the only difference between them is their respective meanings.

## 3. Solution of equation

We assume that at the initial stage we have

$$[Q_\alpha(0), P_\alpha(0)] = [q(0), p(0)] = \mathrm{i}\hbar,$$

where $q(0)$, $p(0)$, $Q_\alpha(0)$ and $P_\alpha(0)$ are the initial values of these operators, in the Heisenberg picture. According to the quantization principle, the commutation relation between $q(t)$ and $p(t)$ or $Q_\alpha(t)$ and $P_\alpha(t)$ is

$$[Q_\alpha(t), P_\alpha(t)] = [q(t), p(t)] = \mathrm{i}\hbar. \tag{8}$$

Using $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_a^{b(t)} f(x,t)\mathrm{d}x = \int_a^{b(t)} \dot{f}(x,t)\mathrm{d}x + f[b(t),t] \times \dfrac{\mathrm{d}b(t)}{\mathrm{d}t}$ and initial values, we have obtained[16]

$$\begin{aligned} Q_\alpha(t) =& Q_\alpha(0)\cos\omega_\alpha t + P_\alpha(0)\omega_\alpha^{-1}\sin\omega_\alpha t \\ &+ c_\alpha \int_0^t \cos\omega_\alpha(t-s)\cdot q(s)\mathrm{d}s. \end{aligned} \tag{9}$$

Supposing $c_\alpha$ is a constant, we have

$$\begin{aligned} \sum_\alpha c_\alpha Q_\alpha =& \sum_\alpha c_\alpha [Q_\alpha(0)\cos\omega_\alpha t + \omega_\alpha^{-1}P_\alpha(0)\sin\omega_\alpha t] \\ &+ c_\alpha^2 \int_0^t \left[\sum_\alpha \cos\omega_\alpha(t-s)\right]\cdot q(s)\mathrm{d}s, \end{aligned}$$

$$\sum_\alpha \cos\omega_\alpha(t-s) = \sum_\alpha \mathrm{Re}\{\exp[-\mathrm{i}\omega_\alpha(t-s)]\} = \delta(t-s),$$

and

$$\sum_\alpha c_\alpha \dot{Q}_\alpha = \sum_\alpha c_\alpha[-\omega_\alpha Q_\alpha(0)\sin\omega_\alpha t + P_\alpha(0)\cos\omega_\alpha t] + c_\alpha^2 \dot{q}(t).$$

Substituting the previous results into Eq.(4a), we obtain

$$\ddot{q} + 2r\dot{q} + \omega_0^2 q = L^{-1}c_\alpha \sum_\alpha [\omega_\alpha Q_\alpha(0)\sin\omega_\alpha t - P_\alpha(0)\cos\omega_\alpha t], \tag{10}$$

where $r = (2L)^{-1}c_\alpha^2$. Using the Laplace transform, we may solve Eq.(10) and find

$$q(t) = q(0)x_1(t) + L^{-1}p(0)x_2(t) + L^{-1}c_\alpha \sum_\alpha [Q_\alpha(0)y_1(t) + P_\alpha(0)y_2(t)], \tag{11a}$$

$$p(t) = Lq(0)\dot{x}_1(t) + p(0)\dot{x}_2(t) + c_\alpha \sum_\alpha [Q_\alpha(0)\dot{y}_1(t) + P_\alpha(0)\dot{y}_2(t)]. \tag{11b}$$

In the above formulae

$$x_1(t) = \exp(-rt)[\cos\Omega t + r\Omega^{-1}\sin\Omega t],$$

$$x_2(t) = \Omega^{-1}\exp(-rt)\sin\Omega t,$$

$$y_1(t) = Y\{\omega_\alpha^2[2r\cos\Omega t + \Omega^{-1}(r^2 - \Omega^2 + \omega_\alpha^2)\sin\Omega t]\exp(-rt) - \omega_\alpha[2r\omega_\alpha\cos\omega_\alpha t - (\omega_0^2 - \omega_\alpha^2)\sin\omega_\alpha t]\},$$

$$y_2(t) = Y\{[(\omega_0^2 - \omega_\alpha^2)\cos\Omega t + r\Omega^{-1}(\omega_0^2 + \omega_\alpha^2)\sin\Omega t]\exp(-rt) + [(\omega_0^2 - \omega_\alpha^2)\cos\omega_\alpha t + 2r\omega_\alpha\sin\omega_\alpha t]\},$$

and

$$Y = [(\omega_0^2 - \omega_\alpha^2)^2 + 4r^2\omega_\alpha^2]^{-1},\ \Omega^2 = \omega_0^2 - r^2 > 0.$$

In [17] Peng pointed out that for a damped harmonic oscillator, because of dissipation, its coordinate $q(t)$ and momentum $p(t)$ are not a pair of linear Hermitian operators. But, in our work, it is easy to confirm that the solutions of the equation satisfy the commutation relation $[Q_\alpha(t), P_\alpha(t)] = [q(t), p(t)] = \mathrm{i}\hbar$. That is to say, when the interactions between electrons and phonons are taken into consideration, the coordinate $q(t)$ and momentum $p(t)$ are, in fact, a pair of linear Hermitian operators. When the crystal lattice oscillation energy is neglected, i.e. $Q_\alpha(0) = P_\alpha(0) = 0$, we have $[q(t), p(t)] = [q(0), p(0)]\exp(-2rt)$. This shows that in a dissipative system the interactions between the electrons and lattice cannot be neglected.

At $T$=0K, we assume that the system is initially in a thermal equilibrium, and all modes of the phonons are in a vacuum state. Then

$$\overline{Q_\alpha(0)} = \overline{P_\alpha(0)} = 0,$$

$$\overline{Q_\alpha^2(0)} = \frac{\hbar}{2\omega_\alpha}, \quad \overline{P_\alpha^2(0)} = \frac{\hbar\omega_\alpha}{2} \quad \alpha = 1, 2, 3, \cdots.$$

With Eqs.(11) we can calculate the average and the square average of charge and current of a mesoscopic $RLC$ circuit. These results are[15]

$$\overline{q(t)} = \overline{q(0)}\cdot x_1(t) + L^{-1}\overline{p(0)}\cdot x_2(t), \tag{12a}$$

$$\overline{p(t)} = L\overline{q(0)}\cdot\dot{x}_1(t) + \overline{p(0)}\cdot\dot{x}_2(t). \tag{12b}$$

$$\overline{q^2(t)} = \overline{q^2(0)}\cdot x_1^2(t) + L^{-2}\overline{p^2(0)}\cdot x_2^2(t) + L^{-1}\overline{[q(0)p(0) + p(0)q(0)]}\cdot x_1(t)x_2(t) + F_1, \tag{13a}$$

$$\overline{p^2(t)} = L^2\overline{q^2(0)}\cdot\dot{x}_1^2(t) + \overline{p^2(0)}\cdot\dot{x}_2^2(t) + L\overline{[q(0)p(0) + p(0)q(0)]}\cdot\dot{x}_1(t)\dot{x}_2(t) + F_2, \tag{13b}$$

where

$$F_1 = \frac{r\hbar}{\pi L}\int_0^\infty \omega_\alpha[\omega_\alpha^{-2}y_1^2(t) + y_2^2(t)]\mathrm{d}\omega_\alpha,$$

$$F_2 = \frac{r\hbar L}{\pi}\int_0^\infty \omega_\alpha[\omega_\alpha^{-2}\dot{y}_1^2(t) + \dot{y}_2^2(t)]\mathrm{d}\omega_\alpha.$$

The above results show that the average and the square average of both charge and current are time-dependent. In the stable state, i.e. $t \to \infty$, the quantum fluctuations of charge and current are

$$\overline{(\Delta q)^2} = F_1(t\to\infty) = \frac{\hbar}{\pi L\Omega}\mathrm{arc\ tan}\left(\frac{\Omega}{r}\right), \tag{14a}$$

$$\overline{(\Delta p)^2} = F_2(t\to\infty) = \frac{\hbar L\omega_0^2}{\pi\Omega}\mathrm{arc\ tan}\left(\frac{\Omega}{r}\right). \tag{14b}$$

From Eqs.(14), when $\omega_0 \gg r$, we have

$$\overline{(\Delta q)^2} = \frac{\hbar}{2L\omega_0}, \quad \overline{(\Delta p)^2} = \frac{\hbar L\omega_0}{2},$$

which are identical with the results in Ref.[18].

## 4. Conclusion

In summary, it is worth taking into consideration the crystal lattice oscillation energy and the interacting energy between electrons and phonons in studying the quantum properties of a dissipative mesoscopic circuit. When the interaction is considered, $q(t)$ and $p(t)$ become a pair of linear Hermitian operators. On this basis, we have studied the time evolution of the average and square average of the charge and current. In a stable state, the quantum fluctuations of the charge and current are not related to the initial state of the system.

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