Vol 11 No 12, December 2002
1009-1963/2002/11(12)/1280-06

Chinese Physics

© 2002 Chin. Phys. Soc.
and IOP Publishing Ltd

Social influence in small-world networks*

Sun Kai(孙 锴), Mao Xiao-Ming(毛晓明), and Ouyang Qi(欧阳颀)†

Department of Physics, Mesoscopic physics Laboratory, Peking University, Beijing 100871, *China*

(Received 3 April 2002; revised manuscript received 12 July 2002)

We report on our numerical studies of the Axelrod model for social influence in small-world networks. Our simulation results show that the topology of the network has a crucial effect on the evolution of cultures. As the randomness of the network increases, the system undergoes a transition from a highly fragmented phase to a uniform phase. We also find that the power-law distribution at the transition point, reported by Castellano *et al*, is not a critical phenomenon; it exists not only at the onset of transition but also for almost any control parameters. All these power-law distributions are stable against perturbations. A mean-field theory is developed to explain these phenomena.

Keywords: social systems, phase transition, self-organization
PACC: 6460C, 0565, 6120J

1. Introduction

Social evolution contains complex phenomena such as self-organization, cooperation, adaptation and termination. These phenomena share common features observed in other natural or man-made complex systems, such as biology, ecology, economy, and the Internet. The study of social evolution has been the aim of many researchers for years, and some interesting findings have been documented in this field.[1−3] Axelrod's model for social influence is important among these.[4] It is used to simulate the evolution of individual attributes, such as language, art, technical standards and social norms that can be changed by means of intercommunications. In his model, prejudice and the influence of media are ignored. The dynamical behaviour of members in a society is supposed to follow two simple disciplines: firstly, a person has a better chance to interact with others who share more identical attributes with him or her; and secondly, the more interaction between two people, the more attributes they are going to share. Under these simple disciplines, the system shows rich dynamical behaviours.

Recently, Castellano *et al* studied this model in a two-dimensional (2D) regular network with computer simulations.[5] They found that, as the initial condition of a system changed, a nonequilibrium phase transition takes place between two totally different asymptotical states: an ordered (uniform) state and a disordered (highly fragmented) state. When the initial culture diversity is not very large, the system would evolve into a uniform state, where almost all the members of the system would share the same social attributes. On the other hand, when the diversity of the initial cultures is increased, a phase transition to a highly fragmented state would be observed. In this state, many different cultures coexist in the system and no one would defeat others. The system could finally have highly diverse cultures.

However, a real social network can never be a regular network, and many researchers have indicated that social networks may be small-world networks.[6] A small-world network lies between a completely regular network and a completely random network.[6] It can be constructed by adding random links to a regular network. Watts and Strogatz discovered that, as soon as a few random links are added to a regular network, the average distance, the least number of edges that constitutes a path between two vertices, decreases very rapidly and the ability of the network to transfer information increases dramatically.

In this paper, we present our study of the influence of the network topology on Axelrod's model. Our study shows that a social system tends to evolve into a uniform state if its members are linked with a random or a small-world network; while it tends to evolve into

*Project supported by the National Natural Science Foundation of China (Grant No 19725519) and Chun-Tsung Foundation of Peking University.

†email: qi@mail.phy.pku.edu.cn

a highly fragmented state if linked with a regular network. A system will undergo a phase transition from a highly fragmented state to a uniform state when the randomness of the network increases. We further study the power-law distribution of the regions' sizes at the critical point reported by Castellano *et al.*[5] We find that the power-law distribution is not a critical phenomenon. It exists not only at the critical point, but also with almost any control parameters. All these distributions have the same power index and can endure disturbances. In the last part of this paper, a mean-field theory is developed to partially explain the power-law distributions observed in the computer simulations.

2. Model

The model used by Castellano *et al* is defined on a square lattice of a linear size L. Every site is linked to its nearest four neighbours. Axelrod's rule states that each site (a member of the society) i has F cultural "features"; each feature is presented by an integer variable $\sigma_{i,f}$ $(f = 1, ..., F)$. Cultural difference is given by the different value of $\sigma_{i,f}$. The initial values of the cultural features are randomly selected according to a discrete Poisson distribution with parameter q: $\text{prob}(\sigma_{i,f} = k) = q^k e^{-q}/k!$. Obviously, q is an index of the initial diversity of the cultures in the system. When q is large, the system has a relatively diverse initial culture distribution, whereas the situation is opposite when q is small. The evolution of the system runs according to the following rules. At each time step, a pair of linked sites i and j are randomly selected, then we randomly choose a feature f. If $\sigma_{i,f} = \sigma_{j,f}$, another feature f' will be randomly chosen from those features which have different values between sites i and j, and we set $\sigma_{i,f'} = \sigma_{j,f'}$. When $\sigma_{i,f} \neq \sigma_{j,f}$, nothing happens.[5]

We introduce the effect of the network topology into the evolution model by expanding the regular lattice of the original model to a small-world network of the same size.[6] In order to obtain a small-world network, we start from a regular square lattice of linear size L, then choose n edges randomly from the network and rewire them randomly. When $n = 0$, the network is regular; when $n = 2L(L-1)$, the network is totally random. Between these two ends, it is a small-world network. Defining the distance between two sites as the least number of edges that constitute a path between them, we see that, as many researchers have reported,[6–8] the average distance of the network decreases dramatically as n increases from zero. This means that the information-diffusing ability of the network increases very rapidly as n increases. We thus choose n as a control parameter in the study of the phase transition from a uniform state to a highly fragmented state.

3. Results

When two individuals (sites) interact through a contact (link), the state (features) of the individuals may change according to Axelrod's rule described in section 2. As time goes by, the individuals at the two ends of a link can only be settled in one of two asymptotical steady states: all the features are equal, or all the features are different. The evolution stops when each pair of connected individuals falls into one of these two steady states. By defining a region as a set of connected sets of sites where all individuals in a region have the same cultural features, the original model[5] shows that, as the initial culture diversity q varies across a certain critical value, a phase transition between a uniform state and a highly fragmented state takes place. This transition can be represented by a sudden change of the average size of the largest region $\langle s_{\max} \rangle$ as a function of the initial culture diversity q.[5] We find that the same phase transition can also be induced when the topology of the network is changed. As shown in Figs.1(a), (b) and (c), $\langle s_{\max} \rangle / L^2$ jumps from almost zero to about 1.0 as the randomness n of the network increases across a critical value n_c, depending on the initial culture diversity q. In the same way, n also influences the critical value of the initial culture diversity q_c. With the increase of n, the critical point (q_c) separating the two phases appears, at larger q, as shown in Figs.1(d), (e) and (f).

Figure 2 shows the phase diagram of society with control parameters q and n. Each point in the diagram is determined by averaging five times the results of computer simulations. The upper part of the diagram is the highly fragmented phase and the lower part is the uniform phase. We observe that the onset of transition moves to higher values of randomness of the network as the initial culture diversity increases. When the society is very diversified ($q > 1200$), even a totally random network ($n = 19800$) cannot unify the society. This reflects the limit of the effect of small-world networks on social diversity.

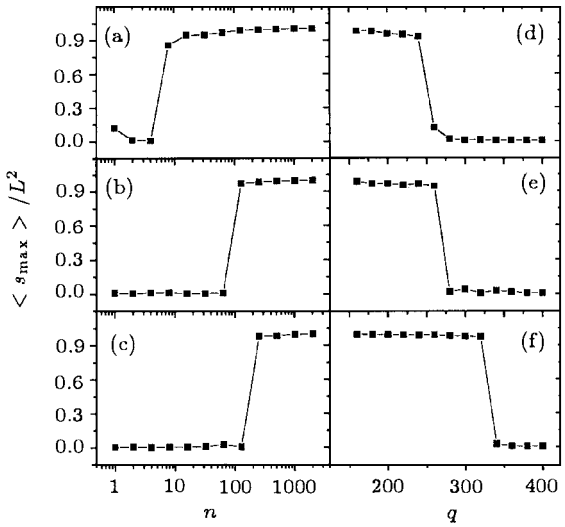


Fig.1. The phase transition as a function of n ((a)–(c)) and q ((d)–(f)). The values of q are (a) 260, (b) 320 and (c) 380. The values of n are (d) 1, (e) 16 and (f) 128. Other control parameters are fixed: $L = 100$ and $F = 10$.

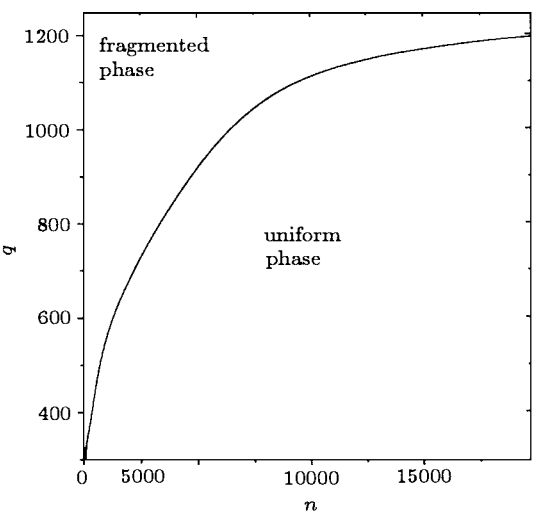


Fig.2. The phase diagram as a function of q and n with $L = 100$ and $F = 10$. The upper part of the figure is the highly fragmented phase and the lower part is the uniform phase.

As Castellano *et al*[5] reported in an earlier study, if we define $U(s)$ as the number of cultural regions whose size is s or larger, then a power-law distribution of the size of the regions can be obtained at the critical point

$$U(s) = (s/s_0)^{1-\gamma}, \qquad (1)$$

where s_0 is a constant. Our study reveals new properties. Firstly, the power-law distribution exists not only at the phase transition point, but also in almost all the fragmented states. For some values of control parameters, it can also be found in a uniform state. Our simulation results are summarized in Fig.3. For the chosen control parameters, the onset of the phase transition takes place when q is increased across 260. We observe that the power-law distributions have the same power-index γ (about 2.7 ± 0.1), but s_0 increases gradually with the increase of q. When q increases to the critical value q_c, s_0 stops increasing. So, in the highly fragmented phase the distributions unify. The result is similar to that in a small-world or random network and the power index γ is also about 2.7 (not shown). The power-law distributions are stable under perturbations. In the simulation, we perturb an asymptotical state by randomly changing 20% of the features of all sites. Each changed feature takes a new number according to the same discrete Poisson distribution as the initial state. Figure 4 gives two examples in this study. In one case (a) the system is located near the onset, in the other (b) it is in the highly fragmented phase. We see that the perturbation seriously drives the distributions out of the asymptotical states. However, as the system evolves to the steady state, the size distribution comes back to almost the same as before the perturbation; the power index γ goes back and s_0 changes only slightly. This proves that the power-law distribution of the size of regions is stable.

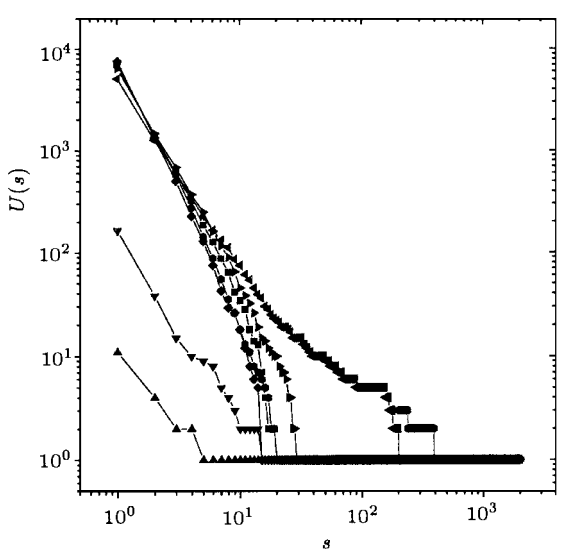


Fig.3. Cumulated size distribution $U(s)$ calculated with different values of q: $\triangle$, $q = 100$; $\bigtriangledown$, $q = 200$; $\lhd$, $q = 300$; $\rhd$, $q = 400$; $\square$, $q = 500$; $\odot$, $q = 600$; $\diamond$, $q = 700$. The transition point is $q_c = 260$. The power index of these distributions is $\gamma \simeq 2.7 \pm 0.1$. Other control parameters are fixed: $L = 100$, $F = 10$, $n = 0$.

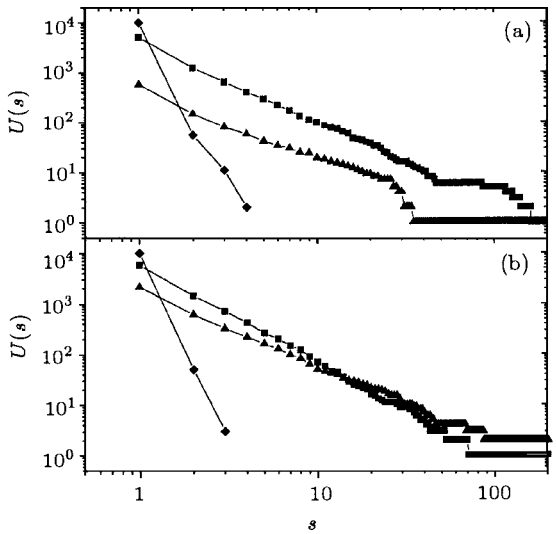


Fig.4. The stability of the power-law size distributions against perturbations: (a) the system is near the onset ($q = 260$); (b) the system is away from the onset ($q = 300$). □, size distribution before the perturbation; ◇, size distribution immediately after the perturbation; △, size distribution at new asymptotic state. Other control parameters are fixed: $L = 100$, $F = 10$, $n = 0$.

4. Discussion and conclusion

The results of our investigation show that the topology of the communicating network in a society plays a crucial role in a social evolution. The observed phenomena can be explained in the context of the information-diffusing ability of a network. In a small-world network, the average distance between two sites drops very rapidly as n increases from zero.[6] So, with a few random links in a regular network, the information of the cultural features will be easier to transport among different regions. This makes it easier for individuals in the system to influence other members and to be influenced by others. In other words, the random links, which represent the cultural influence among distant areas, can make communication easier and make it easier to turn the world into a uniform state. Thus, when we apply the Axelrod's model to study the evolution of a society, the information-diffusing ability of a network must be taken into account, especially when the conditions are near the onset of phase transition. In this situation, a very small error in estimating the network's topology may cause a totally wrong result. On the other hand, the effect of small-world networks has its limit. As shown in Fig.2, q_c remains almost constant as a network is nearly a random network. This can also be explained by the information-diffusing ability of a small-world network. When n increases near the maximum, the decrease of the average distance slows down and almost stops,[6], as does the increase of the information-diffusing ability.

The evolution of the size of regions can be analysed using a mean-field approach. Firstly, we distinguish three different kinds of links: the inner links, which connect the sites in the same region; the living outer links, which connect sites of different regions sharing some identical features; and the dead outer links, which connect sites of different regions sharing no identical feature. Both the living outer links and the dead outer links are called outer links. Then Axelrod's model can be approximately translated as follows. A link is randomly selected. If it is a living outer link, then after a period of evolution, the two regions linked by it will combine into a new region as large as the sum of them. If, on the other hand, it is an inner link or dead outer link, then nothing happens. The procedure goes on again and again until the living outer links are used up. In our mean-field point of view, the selection of a living outer link can be further simplified. We see that the probability of a region being selected and combined with another region is directly proportional to the number of its living outer links, and the number of its living outer links is directly proportional to the number of its total outer links. Thus, we can use the number of outer links that a region has to reflect the probability it will be chosen. So this model can be simplified as: at each period of evolution, two regions are randomly selected according to the numbers of their outer links, and both would disappear with the formation of a larger region. We define $N(s)$ as the number of regions with size s, and $K(s)$ as the average number of outer links (both living and dead links) of a region with size s. When s is small, we assume that the region is linear, so that a region with size s has approximately $s - 1$ inner links. Thus $K(s)$ can be written as $K(s) = 4 \times s - 2 \times (s - 1) = 2 \times s + 2$. With these assumptions, we can write a mean-field evolution equation of $N(s, \tau)$

$$\frac{\mathrm{d}N(s,\tau)}{\mathrm{d}\tau} = -2N(s,\tau)K(s) + \sum_{i=1}^{s-1} N(i,\tau) \times K(i)N(s-i,\tau)K(s-i), \quad (2)$$

$$N(s,0) = L^2\delta_{s,1}, \quad (3)$$

where τ has a timescale of the average interval between two fusion process. The first term on the right-hand side of the evolution equation corresponds to the disappearance of regions with size s, and the second term represents the formation of a region from the disappearance of two smaller regions. The initial condition is chosen such that every individual in the system has different cultural features. Notice that this equation only describes the growth part of the evolution, the mechanism of stopping the growth is not included. In our calculation, we use a reference point $(N(1))$ from simulations to determine when to stop calculation.

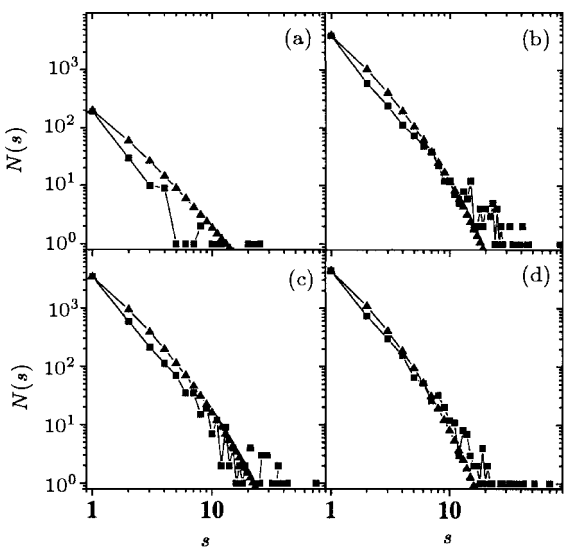


Fig.5. Comparison of Axelrod's model and the mean-field approach. In Axelrod's model, the values of q are (from (a) to (d)): $q = 220, 260, 280, 300$. $L = 100$, $F = 10$ and $n = 0$. The square ($\Box$) represents the size distribution simulated with Axelrod's model; delta ($\triangle$) represents the size distribution calculated using the mean-field theory.

The numerical solution of this dynamic equation is shown in Fig.5. We have compared it with the size distributions in Axelrod's model. Figure 5(a) is in a uniform state, (b) is near the critical point, (c) and (d) are in highly fragmental states. We see that the same dynamic equation can give correct size distributions in different regions of Axelrod's model. The agreement is good when s is small and when $N(s)$ is large; it is less satisfactory when s is large or when $N(s)$ is small. The error is mainly due to the approximation $K(s) = 2 \times s + 2$. This approximation is valid only for small s. When s is large, the number of inner links will be larger than $s - 1$, hence $K(s)$ will seriously deviate from $2 \times s + 2$. Besides, $N(s)$ in the equation represents the probability of a region with a size s. When $N(s)$ is large, using $N(s)$ to describe the number of regions with size s is a good approximation. But when $N(s)$ is comparable to or even smaller than 1, $N(s)$ will lose this meaning. This analysis can also explain the discrepancy shown in Fig.5(a). In a uniform phase, there is a smaller $N(s)$ and a larger s.

Notice that the initial diversity q and the randomness of the networks n do not appear in the mean-field approximation. These two control parameters determine primarily the timescale of the evolution and the time when the evolution stops. As discussed earlier, the first factor is implicitly included in the equation; the second factor is injected through the reference point $(N(1))$ from the result of simulations. The result of our calculations show that the power index γ changes very little in a wide range of $N(1)$. This explains why a power-law distribution can be observed with almost any n and q and with almost the same power index.

The dynamics of a power-law distribution of Axelrod's model can be directly compared with the growth of scale-free networks,[9] which have been found in the World-Wide-Web system[10] and many other social systems. The formation of a scale-free network needs two ingredients:

(a) growth—starting with a small number of sites and at every time step, a new site is added to the system;

(b) preferential attachment—the probability that a new site will be connected to an existing site i is proportional to the number of links this site has.

With these two ingredients, a power-law distribution of the number of links can be predicted.[9] We see that Axelrod's model possesses a similar mechanism. Firstly, although the total number of sites does not increase, some regions do grow during the evolution. Secondly, a large region has more outer links, so it has a better chance to grow. We thus conclude that the power-law distribution observed in Axelrod's model is not a critical phenomenon. Rather, it reflects a special dynamic process found in many systems.[9,10]

Acknowledgment

We thank X. Chen for a helpful discussion.

References

[1] Humberman B A and Lukose R M 1997 *Science* **277** 535

[2] Nowak M A and Sigmund K 1998 *Nature* **393** 573

[3] Puczuski M, Bassler K E and Corral Á 2000 *Phys. Rev. Lett.* **84** 3185

[4] Axelrod R 1997 *The Complexity of Cooperation* (Princeton: Princeton University Press)

[5] Castellano C, Marsili M and Vespignani A 2000 *Phys. Rev. Lett.* **85** 3536

[6] Watts D J and Strogatz S H 1998 *Nature* **393** 440

[7] Kleinberg J M 2000 *Nature* **406** 845

[8] Sun K and Ouyang Q 2001 *Chin. Phys. Lett* **18** 452

[9] Barabási A L, Albert R and Jeong H 1999 *Physica* A **272** 173

[10] Albert R, Jeong H and Barabási A L 1999 *Nature* **401** 130