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Adaptive backstepping control of the uncertain Lü system*

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In this paper we investigate further the method for controlling the uncertain Lü system via adaptive backstepping. The Lü system is transformed into the so-called general strict-feedback form. Then a novel control method is designed for controlling the uncertain Lü system. Especially, this method can overcome the singularity problem. Numerical simulations show the effectiveness and feasibility of this approach.

Keywords: uncertain Lü system, adaptive backstepping, chaotic control
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1. Introduction

Over the last three decades, chaos has been extensively studied within the engineering, scientific and mathematical communities.[1,2] Chaos has been found to be useful, or has a great potential to be useful, in many disciplines such as high-performance circuits and devices, collapse prevention of power systems, information processing, etc.[1,2]

During the past decades, there has been increasing interest in the study of controlling chaotic systems.[1−11] Many different methods have been proposed to achieve chaotic control, such as Ott–Grebogi–Yorke (OGY) methods,[3] the differential geometric method,[4] linear state-space feedback,[5,8] and backstepping design.[6]

However, for some uncertain systems, many of the aforementioned methods will fail. In this paper, we further investigate the method for controlling the uncertain Lü system. It is noticed that a complete input-state linearization leads to the singularity problem. To overcome this problem, researchers have proposed many methods, such as the variable structure method,[12] the adaptive control method,[9−11,16,17] etc. Since the Lü system cannot be controlled directly using the backstepping method for its singularity problem, we transform it into the so-called general strict-feedback form. Then an adaptive controller is presented for controlling the uncertain Lü system. Especially, the designed controller is singularity free, and the closed-loop system is globally stable. Finally, numerical simulations are provided to show the effectiveness of the developed controller.

2. System description

In 1963, Lorenz[13] found the first canonical chaotic attractor. In 1999, Chen[14] found another chaotic attractor. It is noticed that the Lorenz and Chen systems are classified as two *dual dynamical systems* via a crucial condition set by Vaněček and Čelikovský: if these two systems are separated into a linear plus a nonlinear part, then the linear part, represented by a constant matrix $A = \left[a_{ij}\right]_{3\times 3}$, of the Lorenz system satisfies $a_{12}a_{21} > 0$ while that of Chen's system satisfies $a_{12}a_{21} < 0$. Recently, Lü has found a new chaotic system,[15] which satisfies the condition $a_{12}a_{21} = 0$, thereby bridging the gap between the Lorenz and Chen systems.

The nonlinear differential equations that describe the Lü system are

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = -x_1x_3 + cx_2, \\ \dot{x}_3 = x_1x_2 - bx_3, \end{cases} \tag{1}$$

which have a chaotic attractor, as shown in Fig.1, when $a = 36$, $b = 3$, $c = 20$.

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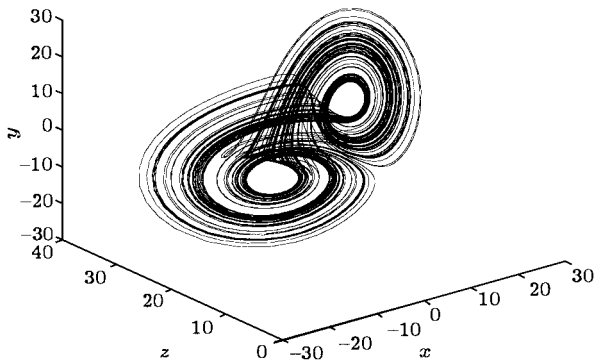


Fig.1. Lü chaotic attractor: $a = 36$, $b = 3$, $c = 20$.

3. The general strict-feedback normal form

From the control engineering and the dynamics points of view, the chaotic Lü attractor is very interesting since it is a critical system. We are interested in developing control techniques for controlling the uncertain Lü attractor.

We assume that the parameters a, b and c of system (1) are unknown. Besides, we add one control input u to the third equation. Thus the controlled system becomes

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = -x_1 x_3 + c x_2, \\ \dot{x}_3 = x_1 x_2 - b x_3 + u\,. \end{cases} \tag{2}$$

Under the assumption that a, b and c are unknown, the controlled Lü system (2) can be easily transformed into the general strict-feedback form with $n = 3$

$$\begin{aligned} &\dot{x}_i = b_i g_i(\bar{x}_i) x_{i+1} + \theta^T F_i(\bar{x}_i) + f_i(\bar{x}_i) \\ &i = 1, 2, \cdots, n-1, \\ &\dot{x}_n = b_n g_n(\bar{x}_n) u + \theta^T F_n(\bar{x}_n) + f_i(\bar{x}_n), \\ &y = x_1, \end{aligned} \tag{3}$$

where $\bar{x}_i = (x_1, \cdots, x_i)^T \in R^i$, $i = 1, 2, \cdots, n$, $u \in R$ and $y \in R$ are the states, input and output respectively; $b = (b_1, b_2, \cdots, b_n)^T \in R^n$ and $\theta \in R^p$ are the vectors of the unknown constant parameters of interest; $g_i(\bar{x}_i)$, $F_i(\bar{x}_i)$, $f_i(\bar{x}_i)$ $i = 1, 2, \cdots, n$ are known smooth nonlinear functions.

Compared with the general strict-feedback form (3), for the controlled Lü system (2), we have

$$\begin{aligned} &g_1(x_1) = 1, && g_2(x_1, x_2) = x_1, && g_3(x_1, x_2, x_3) = 1, \\ &f_1(x_1) = 0, && f_2(x_1, x_2) = 0, && f_3(x_1, x_2, x_3) = x_1 x_2, \\ &F_1(x_1) = (x_1, 0, 0)^T, && F_2(x_1, x_2) = (0, x_2, 0)^T, && F_3(x_1, x_2, x_3) = (0, 0, x_3)^T, \end{aligned} \tag{4}$$

and

$$\begin{aligned} b &= (b_1, b_2, b_3)^T = (a, -1, 1)^T, \\ \theta &= (\theta_1, \theta_2, \theta_3)^T = (-a, c, -b)^T\,. \end{aligned} \tag{5}$$

When all the system parameters a, b and c are unknown, the parameters θ and b_1 are also unknown.

The backstepping design procedure cannot be directly applied to the general strict-feedback form (3) when there are some $g_i(\bar{x}_i) = 0$, $i \in \{1, 2, \cdots, n\}$; however, for some systems, this procedure can overcome the singularity problem caused by some $g_i(\bar{x}_i) = 0$. For system (2), we have $g_2(x_1, x_2) = x_1$, which can take the value of zero. In this paper, we design an adaptive state-feedback controller for system (2), which can guarantee global stability.

4. Adaptive backstepping control

In order to overcome the singularity problem caused by $g_2(x_1, x_2) = x_1$ for the uncertain system (2), we develop a design procedure via backstepping. There are three steps in the backstepping design procedure. At step i, an intermediate control function α_i should be developed using an appropriate Lyapunov function $V_i(t)$. Now we consider the first equation in system (2).

Step 1. Defining $z_1 = x_1$, we have

$$\begin{aligned} \dot{z}_1 = \dot{x}_1 &= a x_2 - a x_1 \\ &= a z_2 + a\alpha_1 - a x_1, \end{aligned} \tag{6}$$

where $z_2 = x_2 - \alpha_1$, with α_1 being an artificial control to be defined later. Using α_1 as a control to stabilize the z_1-subsystem defined by Eq.(6), we can choose the following Lyapunov function:

$$V_1(t) = \frac{1}{2} z_1^2. \tag{7}$$

Calculating the derivative of $V_1(t)$ along system (2), we have

$$\begin{aligned} \dot{V}_1(t) = z_1 \dot{z}_1 &= z_1(a x_2 - a x_1) \\ &= a z_1 z_2 + a z_1 \alpha_1 - a z_1 x_1. \end{aligned} \tag{8}$$

Since $a > 0$ and $x_1 = z_1$, we can choose

$$\alpha_1 = c_0 x_1, \tag{9}$$

where $c_0 < 1$. Letting $c_1 = a(1 - c_0) > 0$, we have

$$\dot{V}_1 = a z_1 z_2 - c_1 z_1^2. \tag{10}$$

We will cancel the first term in the next step. According to Eq.(9), the closed-loop form of Eq.(6) can be written in the form

$$\dot{z}_1 = a z_2 - c_1 z_1. \tag{11}$$

Step 2. In this step, we deal with the singularity problem caused by $-x_1 x_3$ in the second equation of system (2). The derivative of z_2 is expressed as

$$\begin{aligned}\dot{z}_2 =& \dot{x}_2 - \dot{\alpha}_1 \\ =& - x_1 x_3 + c x_2 - c_0(a x_2 - a x_1) \\ =& - x_1 z_3 - x_1 \alpha_2 + c_0 \bar{a} x_1 \\ & + \bar{c}\alpha_1 - c_0 \bar{a}\alpha_1 + (c - c_0 a) z_2 \\ & - c_0(\bar{a} - a) x_1 - (\bar{c} - c)\alpha_1 + c_0(\bar{a} - a)\alpha_1, \end{aligned} \tag{12}$$

where $z_3 = x_3 - \alpha_2$, with α_2 also being an artificial control to be defined later.

We can stabilize the (z_1, z_2)-subsystem defined by Eqs.(11) and (12). In the following we can consider a Lyapunov function $V_2(t)$ defined by

$$V_2(t) = V_1(t) + \frac{1}{2} z_2^2 + \frac{1}{2r}(\bar{a} - a)^2 + \frac{1}{2r}(\bar{c} - c)^2. \tag{13}$$

Its derivative is given by

$$\begin{aligned}\dot{V}_2(t) =& - c_1 z_1^2 + a z_1 z_2 + z_2(\dot{x}_2 - \dot{\alpha}_1) \\ & + \frac{1}{r}(\bar{a} - a)\dot{\bar{a}} + \frac{1}{r}(\bar{c} - c)\dot{\bar{c}} \\ =& - c_1 z_1^2 - x_1 z_2 z_3 + z_2(\bar{a} z_1 - x_1 \alpha_2 + c_0 \bar{a} x_1 \\ & + (\bar{c} - c_0 \bar{a})\alpha_1 + (c - c_0 a) z_2) \\ & + \frac{1}{r}(\bar{a} - a)(\dot{\bar{a}} + r(-c_0 x_1 + c_0 \alpha_1 - z_1) z_2) \\ & + \frac{1}{r}(\bar{c} - c)(\dot{\bar{c}} - r\alpha_1 z_2). \end{aligned} \tag{14}$$

We can eliminate $(\bar{a} - a)$ and $(\bar{c} - c)$ by choosing the following update laws:

$$\begin{cases} \dot{\bar{a}} = r(c_0 x_1 - c_0 \alpha_1 + z_1) z_2, \\ \dot{\bar{c}} = r\alpha_1 z_2. \end{cases} \tag{15}$$

According to Eq.(9) and $z_1 = x_1$, Eq.(14) can be rewritten as

$$\begin{aligned}\dot{V}_2(t) =& - c_1 z_1^2 - x_1 z_2 z_3 + z_2(\bar{a} z_1 - x_1 \alpha_2 + c_0 \bar{a} x_1 \\ & + (\bar{c} - c_0 \bar{a})\alpha_1 + (c - c_0 a) z_2). \end{aligned} \tag{16}$$

Remark: The possibility of $x_1 = 0$ makes α_2 incapable of cancelling the term $(c - c_0 a) z_2$ in Eq.(16). However, we can choose an appropriate c_0 such that $c - c_0 a < 0$. Since $a > c > 0$, it is sufficient to choose $\frac{c}{a} < c_0 < 1$. Letting $c_2 = c_0 a - c$, then we have

$$\begin{aligned} c_1 &= a(1 - c_0) > 0, \\ c_2 &= c_0 a - c > 0. \end{aligned} \tag{17}$$

We choose

$$\alpha_2 = (c_0 + 1)\bar{a} + c_0(\bar{c} - c_0 \bar{a}), \tag{18}$$

then we have

$$\dot{V}_2(t) = - c_1 z_1^2 - c_2 z_2^2 - x_1 z_2 z_3. \tag{19}$$

We will cancel the third term $x_1 z_2 z_3$ in the next step. By using Eq.(18), the closed-loop form (12) can be written in the form

$$\begin{aligned}\dot{z}_2 =& - x_1 z_3 - \bar{a} x_1 - c_0 \bar{c} x_1 + c_0^2 \bar{a} x_1 - c_0(\bar{a} - a) x_1 \\ & + (c - c_0 a) z_2 - c_0 a \alpha_1 + c \alpha_1. \end{aligned} \tag{20}$$

Step 3: The derivative of z_3 can be expressed as

$$\begin{aligned}\dot{z}_3 =& \dot{x}_3 - \dot{\alpha}_2 \\ =& u - b x_3 + x_1 x_2 - \frac{\partial \alpha_2}{\partial \bar{a}}\dot{\bar{a}} - \frac{\partial \alpha_2}{\partial \bar{c}}\dot{\bar{c}}. \end{aligned} \tag{21}$$

In the following we choose an appropriate input u to stabilize the (z_1, z_2, z_3)-subsystem defined by Eqs.(11), (20) and (21). We consider the following Lyapunov function

$$V_3(t) = V_2(t) + \frac{1}{2} z_3^2 + \frac{1}{2r}(\bar{b} - b)^2, \tag{22}$$

then we can obtain the derivative of $V_3(t)$

$$\begin{aligned}\dot{V}_3(t) =& - c_1 z_1^2 - c_2 z_2^2 \\ & + z_3 \Big(- x_1 z_2 + u - \bar{b} x_3 + x_1 x_2 \\ & - \frac{\partial \alpha_2}{\partial \bar{a}}\dot{\bar{a}} - \frac{\partial \alpha_2}{\partial \bar{c}}\dot{\bar{c}} \Big) + \frac{1}{r}(\bar{b} - b)(\dot{\bar{b}} + r z_2 x_3). \end{aligned} \tag{23}$$

By choosing the update law

$$\dot{\bar{b}} = - r z_2 x_3, \tag{24}$$

we eliminate $(\bar{b} - b)$ in Eq.(22).

Let

$$u = - c_3 z_3 + x_1 z_2 + \bar{b} x_3 - x_1 x_2 + \frac{\partial \alpha_2}{\partial \bar{a}}\dot{\bar{a}} + \frac{\partial \alpha_2}{\partial \bar{c}}\dot{\bar{c}}, \tag{25}$$

where $c_3 > 0$. So we have

$$\dot{V}_3(t) = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2, \qquad (26)$$

i.e. $\dot{V}_3(t)$ is negative definite. From the Lasalle–Yoshizawa theorem,[18,19] the equilibrium point $(0, 0, 0)$ of the (z_1, z_2, z_3)-system is globally asymptotically stable, and $(\bar{a}, \bar{b}, \bar{c})$ is globally uniformly bounded. In view of $z_1 = x_1$, $z_2 = x_2 - \alpha_1$ and $\alpha_1 = c_0 x_1$, we know that the states x_1 and x_2 go to zero asymptotically. From $z_3 = x_3 - \alpha_2$ and Eq.(18), we have $x_3 \to (c_0 + 1)\bar{a} + c_0(\bar{c} - c_0\bar{a})$, as $t \to +\infty$, i.e. $x_3(t)$ is bounded. At the same time, from Eq.(25) we can conclude that the control u is also bounded.

5. Numerical simulation

To verify the effectiveness of the proposed design procedure, computer simulations were carried out for the controlled Lü system. We assume that the originally controlled Lü system (2) is in the chaotic state with parameters $a = 36$, $b = 3$ and $c = 20$; the design parameters of controller (25) and parameter update law (15) and (24) are chosen as $c_0 = 0.75$, $c_3 = 48$ and $r = 1$. We choose the initial conditions $x_1(0) = 10$, $x_2(0) = 10$ and $x_3(0) = 10$.

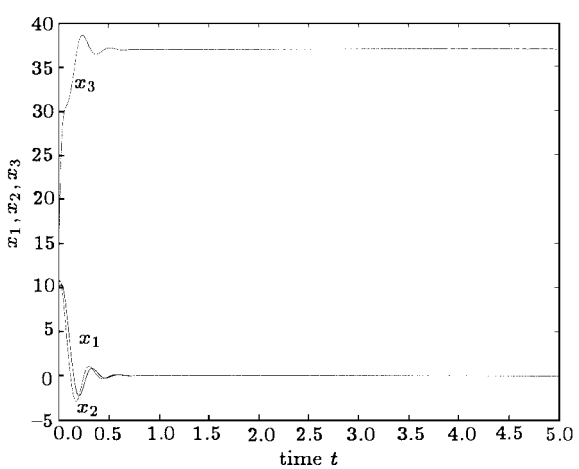


Fig.2. The graph of states x_1, x_2 and x_3 of the controlled Lü system.

Figure 2 shows that the states $x_1(t)$ and $x_2(t)$ of the controlled Lü system (2) are asymptotically regulated to $x_1 = 0$ and $x_2 = 0$, and the state $x_3(t)$ remains bounded.

Figures 3 and 4 show that the parameter estimates $\bar{a}$, $\bar{b}$, $\bar{c}$, and the control u are all bounded.

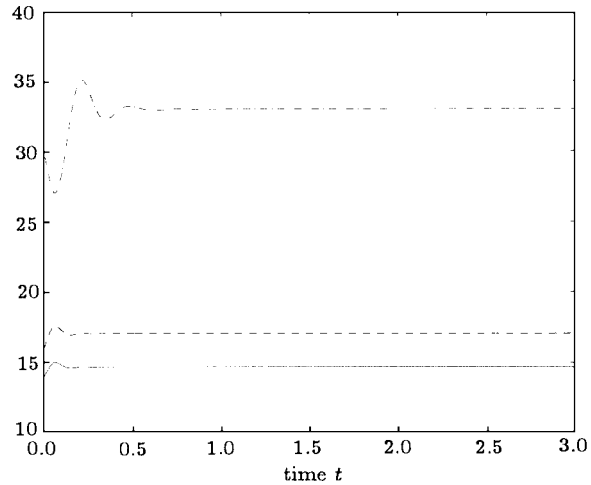


Fig.3. The graph of estimated parameters, $\bar{a}$ (dashed line), $\bar{b}$ (solid line) and $\bar{c}$ (dot-dashed line), of the controlled system.

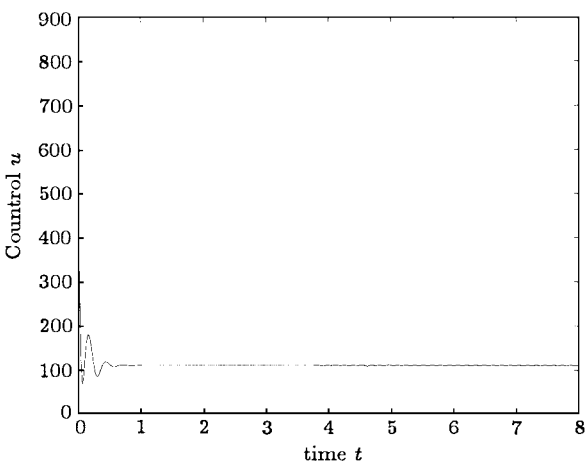


Fig.4. The graph of control action u.

6. Conclusions

We have produced a new and effective control law to control the uncertain Lü system. Using an adaptive backstepping design, we realize the global stability of the uncertain Lü system within finite steps. Compared to other existing chaos control methods, this controller can deal with the uncertain systems and overcome the singularity problem.

It should be noticed that, although this paper focuses mainly on the control of the uncertain Lü attractor, we believe that the higher nonlinearity and complexity of the chaotic Lü attractor can justify the practical application of the proposed method to some other complex uncertain dynamical systems.

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