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Dyadic Green's function for an unbounded gyroelectric chiral medium in cylindrical coordinates

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The dyadic Green's function for an unbounded gyroelectric chiral medium is formulated in the Fourier domain in cylindrical coordinates. After a triple Fourier integral of the dyadic Green's function has been obtained, the integral is reduced by performing the integration over the transverse Fourier variable.

Keywords: gyroelectric chiral medium, dyadic Green's function, Fourier integral, cylindrical coordinates

PACC: 0365D, 0350K, 0340K

1. Introduction

Dyadic Green's functions, as a mathematical kernal or a dielectric medium response, relate directly to the radiated electromagnetic fields and the source distribution. Nowadays, the dyadic Green's function technique is an important method employed elsewhere for electromagnetic waves in various unbounded media and boundary value problems. Although it becomes increasingly difficult to solve the dyadic differential equations governing the Green's functions, an analytic treatment of dyadic Green's functions for various media can be found in many available papers.[1−14] We have listed the works chronologically as Refs.[1–14], but the complete solution is not yet available.

In recent years, there has been a growing interest in the study of electromagnetic wave interactions with complex media. The solutions of Green's functions can be found in terms of eigenfunction representations and spatial representations. The formulations of the Green's functions in gyroelectric chiral media using the eigenfunction expansion technique have been developed by Cheng[16] and Li *et al.*[17,18]

In this paper, Fourier analysis is employed in the formulation of spatial representations of dyadic Green's functions for an unbounded gyroelectric chiral medium, and some ideas in Refs.[14] and [15] will be used. Firstly, the Green's function is expressed as a triple Fourier integral. This result is reduced by performing the integration over the transverse Fourier variable. The formulae in this paper can be applied to several kinds of media including a gyroelectric medium, e.g. a plasma, a uniaxial medium and a gyroelectric chiral medium.

2. Formulation of the problem

A homogeneous gyroelectric chiral medium can be characterized by a set of constitutive relations.[16−18] We select

$$\boldsymbol{D} = \varepsilon_0 \widehat{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E} + i\xi_c \boldsymbol{B}, \boldsymbol{H} = i\xi_c \boldsymbol{E} + \boldsymbol{B}/\mu_0, \quad (1)$$

$$\widehat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \epsilon & -ig & 0 \\ ig & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}. \quad (2)$$

The associated parameters of ϵ, g, ϵ_z, and ξ_c are mentioned in Refs.[16–18]. If a time-dependent $\exp(-\mathrm{i}\omega t)$ is assumed, then the source-incorporated Maxwell's equations lead to

$$\nabla \times \nabla \times \boldsymbol{E}(\boldsymbol{r}) - 2\omega\mu_0\xi_c \nabla \times \boldsymbol{E}(\boldsymbol{r}) - k_0^2 \widehat{\boldsymbol{\varepsilon}} \cdot \boldsymbol{E}(\boldsymbol{r}) = i\omega\mu_0 \boldsymbol{J}(\boldsymbol{r}'), \quad (3)$$

where k_0 is the free-space wavenumber. The electric field can thus be expressed in terms of the dyadic Green's function and electric source distribution as follows

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu_0 \int_{V'} \widehat{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') \mathrm{d}V', \quad (4)$$

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where V' denotes the volume occupied by the exciting current source. If we could compute the dyadic Green's function $\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')$, the problem would be solved. Substituting Eq.(4) into Eq.(3), Eq.(3) can be rewritten as[18]

$$\nabla\times\nabla\times\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')-2\omega\mu_0\xi_c\nabla\times\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')-k_0^2\widehat{\boldsymbol{\varepsilon}}\cdot\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')=\widehat{\boldsymbol{I}}\delta(\boldsymbol{r}-\boldsymbol{r}'),\tag{5}$$

where $\widehat{\boldsymbol{I}}$ denotes the unit dyadic, and $\delta(\boldsymbol{r}-\boldsymbol{r}')$ represents the Dirac δ function. Using Fourier transform, the formal solution of Eq.(5) is written as

$$\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')=\int\int\int\widehat{\boldsymbol{g}}(\boldsymbol{k})\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\mathrm{d}\boldsymbol{k},\tag{6}$$

where $\widehat{\boldsymbol{g}}(\boldsymbol{k})$ is the Fourier transform of $\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')$, and $\boldsymbol{k}=k\hat{k}=p\hat{p}+\lambda\hat{z}$ is the Fourier transform variable in cylindrical coordinates (p,φ_p,λ). The limits of integration in Eq.(6) run from $-\infty$ to $+\infty$ and are omitted for simplicity. This convention will be maintained in the following.

For the computation of the dyadic Green's function in cylindrical coordinates, it is useful to define a matrix

$$\widehat{\boldsymbol{K}}=\begin{bmatrix}0&-\lambda&0\\ \lambda&0&-p\\ 0&p&0\end{bmatrix},\tag{7}$$

so that the dyadic products may be written as a matrix product, e.g.

$$\boldsymbol{k}\times\widehat{\boldsymbol{G}}=\widehat{\boldsymbol{K}}\widehat{\boldsymbol{G}}.\tag{8}$$

Substituting Eq.(6) and the corresponding Fourier integral expression for the delta function into Eq.(5), and applying the operators $\nabla\times\nabla\times$ and $\nabla\times$ to the spatial cylindrical coordinates ρ,φ,z, the matrix equation for the elements of $\widehat{\boldsymbol{g}}(\boldsymbol{k})$ is

$$\widehat{\boldsymbol{A}}(\boldsymbol{k})\cdot\widehat{\boldsymbol{g}}(\boldsymbol{k})=\left(\frac{1}{2\pi}\right)^3\widehat{\boldsymbol{I}},\tag{9}$$

where $$\widehat{\boldsymbol{A}}(\boldsymbol{k})=-\widehat{\boldsymbol{K}}\widehat{\boldsymbol{K}}-i2\omega\mu_0\xi_c\widehat{\boldsymbol{K}}-k_0^2\widehat{\varepsilon}.\tag{10}$$

It is easily seen that

$$\widehat{\boldsymbol{K}}\widehat{\boldsymbol{K}}=k^2(\hat{k}\hat{k}-\widehat{\boldsymbol{I}}).\tag{11}$$

Solving Eq.(9) and substituting this solution into Eq.(6), $\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')$ is obtained in the form of a triple Fourier integral

$$\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')=\frac{1}{(2\pi)^3}\int\int\int\frac{\widehat{\boldsymbol{a}}^{(p)}(\boldsymbol{k})}{D(\boldsymbol{k})}\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')]\mathrm{d}\boldsymbol{k},\tag{12}$$

where $D(\boldsymbol{k})$ is the determinant of $\widehat{\boldsymbol{A}}(\boldsymbol{k})$, and $\widehat{\boldsymbol{a}}^{(p)}(\boldsymbol{k})$ is the adjoint matrix given by

$$\widehat{\boldsymbol{a}}^{(p)}(\boldsymbol{k})=\begin{bmatrix}a_{pp}&a_{p\varphi_p}&a_{p\lambda}\\ a_{\varphi_p p}&a_{\varphi_p\varphi_p}&a_{\varphi_p\lambda}\\ a_{\lambda p}&a_{\lambda\varphi_p}&a_{\lambda\lambda}\end{bmatrix}.\tag{13}$$

The elements of $\widehat{\boldsymbol{a}}^{(p)}(\boldsymbol{k})$ are given as follows:

$$a_{pp}=p^4+[\lambda^2-k_0^2(\epsilon+\epsilon_z)-4\omega^2\mu_0^2\xi_c^2]p^2+k_0^2\epsilon_z(k_0^2\epsilon-\lambda^2),\tag{14}$$

$$a_{p\varphi_p}=-ik_0^2gp^2+ik_0^2\epsilon_z(k_0^2g+2\omega\mu_0\xi_c\lambda),\tag{15}$$

$$a_{p\lambda}=a_{\lambda p}=\lambda p^3-[\lambda(k_0^2\epsilon-\lambda^2)+2\omega\mu_0\xi_c(k_0^2g+2\omega\mu_0\xi_c\lambda)]p,\tag{16}$$

$$a_{\varphi_p p}=ik_0^2gp^2-ik_0^2\epsilon_z(k_0^2\ g+2\omega\mu_0\xi_c\lambda),\tag{17}$$

$$a_{\varphi_p\varphi_p}=-k_0^2\epsilon p^2-k_0^2\epsilon_z(\lambda^2-k_0^2\epsilon),\tag{18}$$

$$a_{\varphi_p\lambda}=ik_0^2(\lambda g+2\omega\mu_0\xi_c\epsilon)p,\tag{19}$$

$$a_{\lambda\varphi_p}=-ik_0^2(\lambda g+2\omega\mu_0\xi_c\epsilon)p,\tag{20}$$

$$a_{\lambda\lambda}=-(k_0^2\epsilon-\lambda^2)p^2+(k_0^2\epsilon-\lambda^2)^2-(k_0^2g+2\omega\mu_0\xi_c\lambda)^2.\tag{21}$$

To study the Green's function, we may regard $\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}')$ as a wave propagating either along the z-axis or away from it. Accordingly, the above integral should be written in a suitable form by expressing $D(\boldsymbol{k})$ as a biquadratic polynomial in the p Fourier variable.

$D(\boldsymbol{k})$ is written as

$$D(\boldsymbol{k})=-k_0^2\epsilon(p^2-p_1^2)(p^2-p_2^2),\tag{22}$$

where $\pm p_i=\pm p_i(\lambda,\varphi_p), i=1,2$ are the four roots of the biquadratic in p

$$-k_0^2\epsilon(p^4+b_pp^2+c_p)=0.\tag{23}$$

Here

$$b_p=-k_0^2[\epsilon_z+(\epsilon^2-g^2)/\epsilon]+4\omega^2\mu^2\xi_c^2/\epsilon+(1+\epsilon_z/\epsilon)\lambda^2,\tag{24}$$

$$c_p=\epsilon_z/\epsilon[(\lambda^2-k_0^2\epsilon)^2-(2\omega\mu\xi_c\lambda+k_0^2g)^2].\tag{25}$$

To eliminate the integration over one Fourier variable, say over p, we use the well-known expansion of a plane wave

$$\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})=\exp(\mathrm{i}\lambda z)\sum_{m=-\infty}^{\infty}i^mJ_m(p\rho)\times\exp[\mathrm{i}m(\varphi-\varphi_p)].\tag{26}$$

Taking into account the above relations, $\hat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')$ is rewritten as

$$\begin{aligned}\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') =& \frac{1}{(2\pi)^3}\int_0^\infty p\mathrm{d}p\int_0^{2\pi}\mathrm{d}\varphi_p\int_{-\infty}^{\infty}\mathrm{d}\lambda \\ &\times\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} i^{m-n}\exp[\mathrm{i}\lambda(z-z')] \\ &\times\exp[\mathrm{i}(m\varphi+n\varphi')]\cdot\exp[-\mathrm{i}(m+n)\varphi_p] \\ &\times\frac{J_m(p\rho)J_n(p\rho')}{D(\boldsymbol{k})}\cdot\widehat{\boldsymbol{a}}(\boldsymbol{k}),\end{aligned}\tag{27}$$

where $\widehat{\boldsymbol{a}}(\boldsymbol{k})$ is the matrix obtained after the translation of $\widehat{\boldsymbol{a}}^{(p)}(\boldsymbol{k})$ from the (p,φ_p,λ) cylindrical coordinate system to the (ρ,φ,z) cylindrical coordinate system, the unit vectors of which do not depend upon the integration variables. This is accomplished via the transformation

$$\widehat{\boldsymbol{a}}(\boldsymbol{k}) = \widehat{\boldsymbol{T}}^{-1}\cdot\widehat{\boldsymbol{a}}^{(p)}(\boldsymbol{k})\cdot\widehat{\boldsymbol{T}}\tag{28}$$

and

$$\widehat{\boldsymbol{T}} = \begin{bmatrix} \hat{p}\cdot\hat{\rho} & \hat{p}\cdot\hat{\varphi} & \hat{p}\cdot\hat{z} \\ \hat{\varphi}_p\cdot\hat{\rho} & \hat{\varphi}_p\cdot\hat{\varphi} & \hat{\varphi}_p\cdot\hat{z} \\ \hat{z}\cdot\hat{\rho} & \hat{z}\cdot\hat{\varphi} & \hat{z}\cdot\hat{z} \end{bmatrix},\tag{29}$$

with $\widehat{\boldsymbol{T}}^{-1} = \widehat{\boldsymbol{T}}^{T}$ due to the orthonormal nature of the cylindrical coordinate system.

For a gyroelectric chiral medium, since Eq.(13) is not concerned with φ_p, the Fourier integral (27) can be simplified as follows:

$$\begin{aligned}\widehat{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') =& \frac{1}{(2\pi)^2}\int_0^\infty p\mathrm{d}p\int_{-\infty}^{\infty}\mathrm{d}\lambda \\ &\times\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} i^{m-n}\exp[\mathrm{i}\lambda(z-z')] \\ &\times\frac{J_m(p\rho)J_n(p\rho')}{D(\boldsymbol{k})}\cdot\widehat{\boldsymbol{a}}(\boldsymbol{k}).\end{aligned}\tag{30}$$

3. Integration over the transverse Fourier variables

In this section, two cases corresponding to $\boldsymbol{r}\neq\boldsymbol{r}'$ and $\boldsymbol{r}=\boldsymbol{r}'$ will be examined, and a study of the integrand function of Eq.(30) is necessary.

As our objective is to perform the integration over variable p, we write $D(\boldsymbol{k})$ as in Eq.(22). Thus, four poles appear in the integrand function, namely, $\pm p_j = \pm p_j(\lambda), j=1,2$. The integrations to be performed are of the form

$$\widehat{\boldsymbol{I}}_p = \int_{-\infty}^{\infty}\frac{\widehat{\boldsymbol{F}}(p,\lambda)\cdot J_m(p\rho)J_n(p\rho')}{(p^2-p_1^2)(p^2-p_2^2)}p\mathrm{d}p.\tag{31}$$

It may easily be verified that the elements of $\widehat{\boldsymbol{F}}(p,\lambda)$ contain p at some power from zero to three except for the $\hat{\rho}\hat{\rho}$ element containing a p^4 term when $\rho=\rho'$. This case corresponds to the singularity source term. The singularity behaviour of Green's dyadic may easily be seen with the similar method used by Cottis *et al.*[14] In the gyrotropic chiral case, the delta-type terms are $(2\epsilon)^{-1}\delta(\boldsymbol{r}-\boldsymbol{r}')(\hat{\rho}\hat{\rho}+\hat{\varphi}\hat{\varphi})$ for a needle-shaped principal volume and ${\epsilon_z}^{-1}\delta(\boldsymbol{r}-\boldsymbol{r}')\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ for a pillbox-shaped one.

The integrals in Eq.(30) are broken into integrals of the form

$$I_p^{(j)} = \int_{-\infty}^{\infty}\frac{p^j\cdot J_m(p\rho)J_n(p\rho')}{(p^2-p_1^2)(p^2-p_2^2)}p\mathrm{d}p, j=0,1,2,3.\tag{32}$$

Using the relations

$$\begin{aligned}&\frac{p}{(p^2-p_1^2)(p^2-p_2^2)} \\ &=\frac{1}{p_1^2-p_2^2}\cdot\left(\frac{p}{p^2-p_1^2}-\frac{p}{p^2-p_2^2}\right),\end{aligned}\tag{33}$$

$$\begin{aligned}&\frac{p^2}{(p^2-p_1^2)(p^2-p_2^2)} \\ &=\frac{1}{p_1^2-p_2^2}\cdot\left(\frac{p^2}{p^2-p_1^2}-\frac{p^2}{p^2-p_2^2}\right),\end{aligned}\tag{34}$$

$$\begin{aligned}&\frac{p^3}{(p^2-p_1^2)(p^2-p_2^2)} \\ &=\frac{1}{p_1^2-p_2^2}\cdot\left(\frac{pp_1^2}{p^2-p_1^2}-\frac{pp_2^2}{p^2-p_2^2}\right),\end{aligned}\tag{35}$$

the following integrals with respect to p are gained

$$\int_{-\infty}^{\infty}\frac{p^j\cdot J_m(p\rho)J_n(p\rho')}{p^2-p_1^2}\mathrm{d}p, j=1,2\tag{36}$$

$$\int_{-\infty}^{\infty}\frac{p^j\cdot J_m(p\rho)J_n(p\rho')}{p^2-p_2^2}\mathrm{d}p, j=1,2.\tag{37}$$

Taking into account the following relations

$$J_m(p\rho) = \frac{1}{2}[H_m^{(1)}(p\rho)+H_m^{(2)}(p\rho)],\tag{38}$$

$$H_m^{(1)}(-p\rho) = (-1)^{n+1}H_m^{(2)}(p\rho),\tag{39}$$

these integrals can be computed. In the case of $\rho>\rho'$, we have

$$\int_{-\infty}^{\infty} \frac{p^j \cdot J_m(p\rho)J_n(p\rho')}{p^2 - p_1^2} \mathrm{d}p = \begin{cases} \dfrac{i\pi}{2} p_1^{j-1} H_m^{(1)}(p_1\rho)[H_m^{(1)}(p_1\rho') + H_m^{(2)}(p_1\rho')], & m+n = \text{even} \\ 0, & m+n = \text{odd} \end{cases} \tag{40}$$

In the case of $\rho < \rho'$, we have

$$\int_{-\infty}^{\infty} \frac{p^j \cdot J_m(p\rho)J_n(p\rho')}{p^2 - p_1^2} \mathrm{d}p = \begin{cases} \dfrac{i\pi}{2} p_1^{j-1} H_m^{(1)}(p_1\rho')[H_m^{(1)}(p_1\rho) + H_m^{(2)}(p_1\rho)], & m+n = \text{even} \\ 0, & m+n = \text{odd} \end{cases} \tag{41}$$

For Eqs.(39) and (40), the integral (36) can be obtained using a similar method.

4. Discussion

We discuss the following three simplified cases:

(a) In the uniaxial chiral case, namely $g = 0$, Eq.(30) can be simplified, but it is still required to solve the integration in which the integrand function contains the Bessel functions.

(b) In the gyroelectric case, namely $\xi_c = 0$, similarly, Eq.(30) is simplified to an integration in which the integrand function contains Bessel functions. The formulae in this paper can be applied by putting $\xi_c = 0$ to the radiation of an electric dipole in the gyroelectric medium, such as plasma, directly.

(c) In the uniaxial case, i.e. $\xi_c = 0$ and $g = 0$, after performing the integration over the transverse Fourier variable p, the solution of Eq.(30) can be obtained by performing the integration over the longitudinal Fourier variable λ. In both cylindrical and spherical coordinates, the Green's functions in a uniaxial medium have been solved analytically many years ago.

5. Conclusion

In this paper, the dyadic Green's function for an unbounded gyroelectric chiral medium has been examined analytically. Following the Fourier analysis, it has been shown that the dyadic Green's function can be expressed as a triple Fourier integral, which can be reduced by performing the integration over the transverse Fourier variable.

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