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Invariant theory and exact solutions of the time-dependent supersymmetric two-level multiphoton Jaynes–Cummings model*

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On the basis of the fact that the two-level multiphoton Jaynes–Cummings (TLMJC) model possesses a supersymmetric structure, an invariant is constructed in terms of the supersymmetric generators by working in the sub-Hilbert-space corresponding to a particular eigenvalue of the conserved supersymmetric generators (the time-independent invariant). In this paper, we investigate the invariant-related unitary transformation approach to exact solutions of the time-dependent TLMJC model.

Keywords: supersymmetric Jaynes–Cummings model, exact solutions, invariant theory, geometric phase factor

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1. Introduction

At present, the problems associated with a two-level atom are attracting much attention from many researchers in various fields, such as quantum optics, quantum information and quantum computation.[1−3] The interaction between a two-level atom and a quantized single-mode electromagnetic field is described by the Jaynes–Cummings (JC) model,[4] which can be applied to the study of many quantum effects such as the quantum collapses and revivals of the atomic inversion, photon antibunching, squeezing of the radiation field, inversionless light amplification as well as electromagnetic induced transparency,[5−9] etc. By making use of the generalized invariant theory,[10] we can obtain exact solutions and geometric phase factors of the two-level JC model whose Hamiltonian parameters are totally time-dependent. In addition, there is another type of JC model (the so-called two-level multiphoton Jaynes–Cummings model (TLMJC)), which possesses supersymmetric Lie-algebraic structure. In this generalization of the JC model, the atomic transitions are mediated by k photons.[11−15] Singh[16] has shown that this model can be used to study multiple atom scattering of radiation and multiphoton emission, absorption, and laser processes. Some authors have introduced a supersymmetric unitary transformation to diagonalize the Hamiltonian of this JC model and to obtain the eigenfunctions of the stationary Schrödinger equation.[17,18] In this paper, we generalize this method and obtain the exact solutions and the expressions for the geometric phase factor of the totally time-dependent TLMJC model through the invariant-related unitary transformation formulation.

The invariant theory that is appropriate to treat the time-dependent systems was first proposed by Lewis and Riesenfeld (LR)[19] in 1969. In 1991, Gao *et al* generalized the LR invariant theory and proposed the invariant-related unitary transformation formulation.[10,20] One of the advantages of this unitary transformation method is that it can transform the hidden form, which is related to the chronological product, of the time-evolution operator $U(t)$ into an explicit expression. Many works have shown that the invariant-related unitary transformation approach is a powerful tool for investigating time-dependent systems and the geometric phase factor.[21−23]

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2. The invariant theory and the invariant-related unitary transformation formulation

In order to illustrate the LR invariant theory,[19] we consider a one-dimensional system whose Hamiltonian $H(t)$ is time-dependent. According to LR invariant theory, a Hermitian operator $I(t)$ is called invariant if it satisfies the following invariant equation

$$\frac{\partial I(t)}{\partial t}+\frac{1}{i}[I(t),H(t)]=0. \tag{1}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n,t\rangle$ is given by

$$I(t)\,|\lambda_n,t\rangle=\lambda_n\,|\lambda_n,t\rangle\,, \tag{2}$$

where

$$\frac{\partial\lambda_n}{\partial t}=0. \tag{3}$$

The time-dependent Schrödinger equation for the system is (in the units $\hbar=1$)

$$i\frac{\partial\,|\,\Psi(t)\rangle_s}{\partial t}=H(t)\,|\,\Psi(t)\rangle_s\,. \tag{4}$$

In terms of the LR invariant theory, the particular solution $|\lambda_n,t\rangle_s$ of Eq.(4) differs from the eigenfunction $|\lambda_n,t\rangle$ of the invariant $I(t)$ only by a phase factor $\exp[i\phi_n(t)]$. Then the general solution of the Schrödinger equation (4) can be written as

$$|\,\Psi(t)\rangle_s=\sum_n C_n\exp[i\phi_n(t)]\,|\lambda_n,t\rangle\,, \tag{5}$$

where

$$\phi_n(t)=\int_0^t\left\langle\lambda_n,t'\left|\,i\frac{\partial}{\partial t'}-H(t')\,\right|\lambda_n,t'\right\rangle\mathrm{d}t',$$
$$C_n=\langle\lambda_n,t=0\,|\,\Psi(t=0)\rangle_s\,. \tag{6}$$

$|\lambda_n,t\rangle_s=\exp[i\phi_n(t)]\,|\lambda_n,t\rangle$ $(n=1,2,\cdots)$ are said to form a complete set of solutions of Eq.(4). The statement outlined above is the basic content of the LR invariant theory.

A time-dependent unitary transformation operator can be constructed to transform $I(t)$ into a time-independent invariant $I_V\equiv V^\dagger(t)I(t)V(t)$ with[10,20]

$$I_V\,|\lambda_n\rangle=\lambda_n\,|\lambda_n\rangle\,, \tag{7}$$

$$|\lambda_n\rangle=V^\dagger(t)\,|\lambda_n,t\rangle\,. \tag{8}$$

Under the unitary transformation $V(t)$, the Hamiltonian $H(t)$ is correspondingly changed into $H_V(t)$

$$H_V(t)=V^\dagger(t)H(t)V(t)-V^\dagger(t)i\frac{\partial V(t)}{\partial t}. \tag{9}$$

In accordance with this unitary transformation method,[10] it is very easy to verify that the particular solution $|\lambda_n,t\rangle_{s0}$ of the time-dependent Schrödinger equation associated with $H_V(t)$,

$$i\frac{\partial\,|\lambda_n,t\rangle_{s0}}{\partial t}=H_V(t)\,|\lambda_n,t\rangle_{s0}\,, \tag{10}$$

is different from the eigenfunction $|\lambda_n\rangle$ of I_V only by the same phase factor $\exp[i\phi_n(t)]$ as that in Eq.(5), i.e.

$$|\lambda_n,t\rangle_{s0}=\exp[i\phi_n(t)]\,|\lambda_n\rangle\,. \tag{11}$$

Substituting $|\lambda_n,t\rangle_{s0}$ of Eq.(10) into Eq.(11) yields

$$-\dot{\phi}(t)\,|\lambda_n\rangle=H_V(t)\,|\lambda_n\rangle\,, \tag{12}$$

indicating that $H_V(t)$ differs from $I_V(t)$ only by a time-dependent multiplying c-number factor. It can be seen from Eq.(12) that the particular solution of Eq.(10) can be easily obtained by calculating the phase from Eq.(12). Thus, we are led to the conclusion that, if $V(t)$, I_V, $H_V(t)$ and the eigenfunction $|\lambda_n\rangle$ of I_V have been found, the problem of solving the complicated time-dependent Schrödinger equation (4) reduces to that of solving the much simplified equation (10). In this paper, we obtain the exact solutions of the time-dependent Schrödinger equation describing the TLMJC model and the expression for its geometric phase factor by making use of this invariant-related unitary transformation method.

3. Exact solutions of the time-dependent TLMJC model

The Hamiltonian of the time-dependent TLMJC model under the rotating wave approximation is given by

$$H(t)=\omega(t)a^\dagger a+\frac{\omega_0(t)}{2}\sigma_z + g(t)(a^\dagger)^k\sigma_-+g^*(t)a^k\sigma_+. \tag{13}$$

Here $a^\dagger$ and a are the creation and annihilation operators for the electromagnetic field, which obey the commutation relation $\left[a,a^\dagger\right]=1$; $\sigma_\pm$ and σ_z denote the two-level atom operators which satisfy the commutation relation $[\sigma_z,\sigma_\pm]=\pm2\sigma_\pm$; $g(t)$ and $g^*(t)$ are the coupling coefficients and k is the photon number in each atom transition process; $\omega_0(t)$ and $\omega(t)$ are the transition frequency and the mode frequency, respectively. All the parameters in $H(t)$ are time-dependent.

The supersymmetric structure can be found in the TLMJC model by defining the following supersymmetric transformation generators:[17,18]

$$N = a^\dagger a + \frac{k-1}{2}\sigma_z + \frac{1}{2}$$
$$= \begin{pmatrix} a^\dagger a + \frac{k}{2} & 0 \\ 0 & aa^\dagger - \frac{k}{2} \end{pmatrix},$$
$$N' = \begin{pmatrix} a^k(a^\dagger)^k & 0 \\ 0 & (a^\dagger)^k a^k \end{pmatrix},$$
$$Q = (a^\dagger)^k\sigma_- = \begin{pmatrix} 0 & 0 \\ (a^\dagger)^k & 0 \end{pmatrix},$$
$$Q^\dagger = a^k\sigma_+ = \begin{pmatrix} 0 & a^k \\ 0 & 0 \end{pmatrix}. \qquad (14)$$

It is easily verified that $(N, N', Q, Q^\dagger)$ form supersymmetric generators and have supersymmetric Lie algebra properties, i.e.,

$$Q^2 = (Q^\dagger)^2 = 0, \quad [Q^\dagger, Q] = N'\sigma_z,$$
$$\left[N, N'\right] = 0, \quad [N, Q] = Q,$$
$$\left[N, Q^\dagger\right] = -Q^\dagger, \quad \left\{Q^\dagger, Q\right\} = N',$$
$$\{Q, \sigma_z\} = \left\{Q^\dagger, \sigma_z\right\} = 0,$$
$$[Q, \sigma_z] = 2Q, \quad \left[Q^\dagger, \sigma_z\right] = -2Q^\dagger,$$
$$\left(Q^\dagger - Q\right)^2 = -N', \qquad (15)$$

where $\{\}$ denotes the anticommuting bracket. By using Eqs.(14) and (15), the Hamiltonian (13) of the TLMJC model can be rewritten as

$$H(t) = \omega(t)N + \frac{\omega(t) - \delta(t)}{2}\sigma_z + g(t)Q + g^*(t)Q^\dagger - \frac{\omega(t)}{2}, \qquad (16)$$

with $\delta(t) = k\omega(t) - \omega_0(t)$.

The equation which governs the time evolution of the TLMJC model is the time-dependent Schrödinger equation (4). In the following, we show the solvability of Eq.(4) by using the generalized invariant formulation.

According to the invariant theory, we should first construct an invariant $I(t)$. From the invariant equation (1) we can see that $I(t)$ is the linear combination of N, σ_z, Q and $Q^\dagger$. However, it should be pointed out that the generalized invariant theory can only be applied to the study of the system for which there exists the quasi-algebra defined in Ref.[24]. It is easily seen from Eq.(15) that there is no such a quasi-algebra for the TLMJC model. We generalize the method which has been used for finding the dynamical algebra $O(4)$ of the hydrogen atom to treat this type of time-dependent model. In the case of hydrogen, the dynamical algebra $O(4)$ was found by working in the sub-Hilbert-space corresponding to a particular eigenvalue of the Hamiltonian.[25] In this paper, we show that a generalized quasi-algebra can also be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue of N' in the time-dependent TLMJC model. This generalized quasi-algebra enables us to obtain the complete set of exact solutions for the Schrödinger equation. It is easily verified that N' commutes with $H(t)$ and is a time-independent invariant according to Eq.(1).

By making use of $a^k(a^\dagger)^k|m\rangle = \frac{(m+k)!}{m!}|m\rangle$ and $(a^\dagger)^k a^k|m+k\rangle = \frac{(m+k)!}{m!}|m+k\rangle$, then we can arrive at

$$N'\begin{pmatrix} |m\rangle \\ |m+k\rangle \end{pmatrix} = \lambda_m \begin{pmatrix} |m\rangle \\ |m+k\rangle \end{pmatrix}, \qquad (17)$$

with $\lambda_m = \frac{(m+k)!}{m!}$. Thus, we obtain the supersymmetric quasi-algebra $(N, Q, Q^\dagger, \sigma_z)$ in the sub-Hilbert-space corresponding to the particular eigenvalue λ_m of N', by replacing the generator N' with λ_m in the commutation relations in (15), i.e.,

$$\left[Q^\dagger, Q\right] = \lambda_m\sigma_z, \quad \left\{Q^\dagger, Q\right\} = \lambda_m,$$
$$\left(Q^\dagger - Q\right)^2 = -\lambda_m. \qquad (18)$$

In accordance with the invariant theory, the invariant $I(t)$ is often of the form

$$I(t) = c(t)Q^\dagger + c^*(t)Q + b(t)\sigma_z, \qquad (19)$$

where $c^*(t)$ is the complex conjugation of $c(t)$, and $b(t)$ is real. Substituting expressions (16) and (19) for $I(t)$ and $H(t)$ into Eq.(1) leads to the following set of auxiliary equations

$$\dot{c} - \frac{1}{i}[c\delta + 2bg] = 0,$$
$$\dot{c}^* + \frac{1}{i}[c^*\delta + 2bg^*] = 0,$$
$$\dot{b} + \frac{1}{i}\lambda_m(c^*g - cg^*) = 0, \qquad (20)$$

where the dot denotes the time derivative. The three time-parameters c, c^* and b in $I(t)$ are determined by these three auxiliary equations.

Using the invariant-related unitary transformation method,[10] we define the unitary transformation operator as follows

$$V(t) = \exp[\beta(t)Q - \beta^*(t)Q^\dagger], \tag{21}$$

with $\beta^*(t)$ being the complex conjugation of $\beta(t)$. With the help of the commutation relations (15), it can be found that, by complicated and lengthy computations, if $\beta(t)$ and $\beta^*(t)$ satisfy the following equations

$$\sin(4\beta\beta^*\lambda_m)^{\frac{1}{2}} = \frac{\lambda_m(c\beta^* + c^*\beta)}{(4\beta\beta^*\lambda_m)^{\frac{1}{2}}},$$
$$\cos(4\beta\beta^*\lambda_m)^{\frac{1}{2}} = b, \tag{22}$$

a time-independent invariant can be obtained as follows

$$I_V \equiv V^\dagger(t)I(t)V(t) = \sigma_z. \tag{23}$$

From Eq.(22), we substitute the time-dependent parameters θ and ϕ for c, c^* and b in $I(t)$ for simplicity and convenience, and the results are

$$\beta = -\frac{\frac{\theta}{2}\exp(-i\phi)}{\lambda_m^{\frac{1}{2}}}, \quad \beta^* = -\frac{\frac{\theta}{2}\exp(i\phi)}{\lambda_m^{\frac{1}{2}}},$$
$$c = -\frac{\sin\theta\exp(-i\phi)}{\lambda_m^{\frac{1}{2}}}, \quad c^* = -\frac{\sin\theta\exp(i\phi)}{\lambda_m^{\frac{1}{2}}}. \tag{24}$$

Thus, the invariant $I(t)$ in Eq.(19) can be rewritten as

$$I(t) = -\frac{\sin\theta}{\lambda_m^{\frac{1}{2}}}[\exp(-i\phi)Q + \exp(i\phi)Q^\dagger] + \cos\theta\sigma_z. \tag{25}$$

Meanwhile, under the unitary transformation (21), the Hamiltonian (16) can be transformed into

$$\begin{aligned} H_V(t) \equiv & V^\dagger(t)H(t)V(t) - V^\dagger(t)i\frac{\partial}{\partial t}V(t) \\ = & \omega N - \frac{\omega}{2} + \left\{\frac{\omega}{2}(1-\cos\theta)\right. \\ & - \frac{1}{2}\lambda_m^{\frac{1}{2}}[g\exp(i\phi) + g^*\exp(-i\phi)]\sin\theta \\ & \left. + \frac{\omega-\delta}{2}\cos\theta - \frac{\dot{\phi}}{2}(1-\cos\theta)\right\}\sigma_z, \end{aligned} \tag{26}$$

where we use the Baker–Campbell–Hausdorff formula[26]

$$\begin{aligned} V^\dagger(t)\frac{\partial}{\partial t}V(t) = & \frac{\partial}{\partial t}L + \frac{1}{2!}\left[\frac{\partial}{\partial t}L, L\right] \\ & + \frac{1}{3!}\left[\left[\frac{\partial}{\partial t}L, L\right], L\right] \\ & + \frac{1}{4!}\left[\left[\left[\frac{\partial}{\partial t}L, L\right], L\right], L\right] + \cdots, \end{aligned} \tag{27}$$

with $V(t) = \exp[L(t)]$. The eigenstates of σ_z corresponding to the eigenvalues $\sigma = +1$ and $\sigma = -1$ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the eigenstate of $N^{'}$ is $\begin{pmatrix} |m\rangle \\ |m+k\rangle \end{pmatrix}$ in terms of Eq.(17). From Eqs.(6), (11) and (12), we obtain two particular solutions of the time-dependent Schrödinger equation of the time-dependent TLMJC model which are written in the form

$$\begin{aligned} |\Psi_{m,\sigma=+1}(t)\rangle = \exp\left\{\frac{1}{i}\int_0^t [\dot{\varphi}_{d,\sigma=+1}(t^{'}) \right. \\ \left. + \dot{\varphi}_{g,\sigma=+1}(t^{'})]\mathrm{d}t^{'}\right\} V(t)\begin{pmatrix} |m\rangle \\ 0 \end{pmatrix}, \end{aligned} \tag{28}$$

with

$$\begin{aligned} \dot{\varphi}_{d,\sigma=+1}(t^{'}) = & \left(m + \frac{k}{2}\right)\omega(t^{'}) - \frac{1}{2}\lambda_m^{\frac{1}{2}}\{g(t^{'})\exp[i\phi(t^{'})] \\ & + g^*(t^{'})\exp[-i\phi(t^{'})]\}\sin\theta(t^{'}) \\ & - \frac{\delta(t^{'})}{2}\cos\theta(t^{'}) \end{aligned} \tag{29}$$

and

$$\dot{\varphi}_{g,\sigma=+1}(t^{'}) = -\frac{\dot{\phi}(t^{'})}{2}[1-\cos\theta(t^{'})]; \tag{30}$$

and

$$\begin{aligned} |\Psi_{m,\sigma=-1}(t)\rangle = \exp\left\{\frac{1}{i}\int_0^t [\dot{\varphi}_{d,\sigma=-1}(t^{'}) \right. \\ \left. + \dot{\varphi}_{g,\sigma=-1}(t^{'})]\mathrm{d}t^{'}\right\} V(t')\begin{pmatrix} 0 \\ |m+k\rangle \end{pmatrix} \end{aligned} \tag{31}$$

with

$$\begin{aligned} \dot{\varphi}_{d,\sigma=-1}(t^{'}) = & (m + \frac{k}{2})\omega(t^{'}) + \frac{1}{2}\lambda_m^{\frac{1}{2}}\{g(t^{'})\exp[i\phi(t^{'})] \\ & + g^*(t^{'})\exp[-i\phi(t^{'})]\}\sin\theta(t^{'}) \\ & + \frac{\delta(t^{'})}{2}\cos\theta(t^{'}) \end{aligned} \tag{32}$$

and

$$\dot{\varphi}_{g,\sigma=-1}(t^{'}) = \frac{\dot{\phi}(t^{'})}{2}[1-\cos\theta(t^{'})]. \tag{33}$$

These two particular solutions of the Schrödinger equation (4) contain the corresponding dynamical phase factor $\exp\left[\frac{1}{i}\int_0^t \dot{\varphi}_{d,\sigma}(t^{'})\mathrm{d}t^{'}\right]$ and the geometric phase factor $\exp\left[\frac{1}{i}\int_0^t \dot{\varphi}_{g,\sigma}(t^{'})\mathrm{d}t^{'}\right]$ with $\sigma = \pm 1$. It

should be noted that when the parameter θ is taken to be a constant in expressions (30) and (33), the geometric phases in a cycle associated with $\phi(t)$ in the parameter space of the invariant $I(t)$ can be rewritten as

$$\phi_\sigma(T) = -\frac{\sigma}{2}2\pi(1-\cos\theta), \tag{34}$$

where $2\pi(1-\cos\theta)$ is the solid angle over the parameter space of the invariant $I(t)$, which represents the geometric meanings of the phase factor. Apparently, it can be seen that the dynamical phase factor is dependent on the transition frequency $\omega_0(t)$ and the mode frequency $\omega(t)$, and the coupling coefficients $g(t)$ and $g^*(t)$ as well, whereas the geometric phase factor is independent of these frequency parameters and the coupling coefficients.

4. Concluding remarks

We have constructed an invariant in the sub-Hilbert-space corresponding to a particular eigenvalue of the time-independent invariant $N^{'}$ and we have obtained the exact solutions of the time-dependent TLMJC model by making use of the invariant-related unitary transformation formulation. In view of the above calculation, we can see that this unitary transformation formulation can have some useful applications, for instance, it can solve the time-dependent systems and treat the geometric phase factor, and obtain the explicit expressions, instead of the hidden form, for the evolution operator of the wavefunctions. This method can also be generalized to study the time-evolutions of the quantum Klein–Gordon field and Dirac field in time-dependent backgrounds, such as spatially homogeneous electric field and Friedmann–Robertson–Walker flat space–times.[27−29]

Since the three-level two-mode JC model plays an important role in quantum optics,[30−35] the supersymmetric structure and the exact solutions of the time-dependent three-level two-mode multiphoton JC model deserves further investigations using the formalism suggested in this paper.

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