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Existence and linear stability of planar waves in the Oregonator model*

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In this paper, various waves in the Oregonator model are investigated in detail. The existence of planar waves (in particular, spiral waves) is rigorously proven and their linear stability is analysed. In addition, some non-planar waves in the model are also presented.

Keywords: excitable media, travelling wave, spiral wave, linear stability
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1. Introduction

The Oregonator model[1,2] originates from the famous Belousov–Zhabotinsky reaction which leads to a variety of non-equilibrium spatial structures. It supports the propagation of single or periodic travelling excitable waves, as well as more complex regimes such as rotating spiral waves and turbulence, and serves as a paradigm for pattern formation in excitable media.[3−5]

The model is generally described by the following two-component reaction-diffusional system

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_1 \Delta u + \dfrac{1}{\varepsilon} f(u,\ v), \\ \dfrac{\partial v}{\partial t} = D_2 \Delta v + g(u,\ v), \end{cases} \tag{1}$$

where $f(u,\ v) = u - u^2 - fv\dfrac{u-\mu}{u+\mu}$, $g(u,\ v) = u - v$. In this equation, the parameters satisfy: $0 < \mu < 1$, $0 < \epsilon < 1$, $0 < f < +\infty$. D_1 and D_2 are diffusing coefficients, Δ is Laplacian, and spatial variables may be of one, two or three dimensions.

In recent years, much effort has been focused on the study of this model.[5−12] However, there have been few theoretical works, and most of the studies are numerical simulations and approximate estimates.

The properties of waves in this model have been studied by many researchers.[4,8,13−16] In spite of this, the important and fundamental problems, such as the existence and stability of waves, have not yet been solved.

To emphasize the importance of our results, it is necessary to review the previously obtained results and to address the difficulties in proving the existence and stability of wave pattern solutions in this field. Keener[8] studied a class of two-component reaction-diffusional equations with excitable dynamics in the framework of singular perturbation theory, and showed how stable travelling pulses and periodic wave trains in one spatial dimension evolve from initial states, and that steady spiral waves can exist in two-dimensional regimes. According to this, some qualitative properties of waves in the Oregonator model can be established. But, we do not know what types of waves may exist and in what range of parameters these waves exist. In addition, Kessler *et al*[13] used the formulated piecewise-linear Oregonator model to study the stability of planar travelling waves to general perturbation. By dividing the regime of existence of the waves into the excited and quiescent parts, they analysed the stability of the waves in so-called Fife regimes. This analysis, however, is restricted only to some special waves. Moreover, the treatment is not always feasible. Because of this, the problem of how to prove the existence and stability of waves in general excitable media is open to question.

In this paper, travelling excitable waves in the Oregonator model are investigated in detail, and some

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rigorous results for the existence and linear stability of the waves are given. Besides, some non-planar waves are also found in the model, for the first time as far as we know.

2. Existence of waves

In this section, various waves for Eq.(1) are investigated and the existence of planar waves in a certain range of parameters is rigorously proven. In addition, the existence of some non-planar waves is shown by expanding the solutions of Eq.(1) into the series of Bessel functions with respect to the wave amplitude, and by analysing the leading terms in the expansion.

To be specific, we introduce some symbols used below. For Eq.(1), the unique positive steady state, denoted by (u_+, v_+), satisfies: $f(u_+, v_+) = 0$, $g(u_+, v_+) = 0$, where $v_+ = u_+ = \dfrac{1-f-\mu+\sqrt{(1-f-\mu)^2+4\mu(1+f)}}{2}$. We denote $a_1 = \dfrac{1}{\varepsilon}\left[1-2u_+-\dfrac{2f\mu u_+}{(u_++\mu)^2}\right]$, $b_1 = -\dfrac{f}{\varepsilon}\left[\dfrac{u_+-\mu}{u_++\mu}\right]$, $P = 1-a_1$, $Q = -(a_1+b_1)$, $\triangle = P^2-4Q$, $X = \begin{pmatrix} u-u_+ \\ v-u_+ \end{pmatrix}$, $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$, $F[X] = \begin{pmatrix} \frac{1}{\varepsilon}f(u-u_+, v-u_+) \\ g(u-u_+, v-u_+) \end{pmatrix}$, $A = \begin{pmatrix} a_1 & b_1 \\ 1 & -1 \end{pmatrix}$. Then Eq.(1) can be rewritten in the following tidy form

$$X_t' = D\Delta X + F[X], \tag{2}$$

whose linearized equation at the steady-state point (u_+, v_+) is

$$X_t' = D\Delta X + AX. \tag{3}$$

Hereafter the subscripted variable is used to denote differentiation with respect to the argument.

2.1. Planar wave

Let $\boldsymbol{z}$ be a spatial coordinate vector and $\xi = t - \dfrac{\boldsymbol{k}z}{\omega_1}$ (where ω_1 is the wave frequency and $\boldsymbol{k}$ is the wave vector). In the following, for the same letter, the bold typeface represents the vector while the normal typeface denotes its norm). Thus, the problem of seeking for the planar wave solution of Eq.(2) can be changed into that of looking for the periodic solution of the following equation with respect to ξ

$$X_\xi' = F[X] + \beta_1 D X_{\xi\xi}'', \tag{4}$$

where $\beta_1 = \left(\dfrac{k}{\omega_1}\right)^2$.

Now, we take β_1 as a parameter in Eq.(4). One of our aims is to find conditions for the existence of periodic solutions of Eq.(4). For this, we let $Y = X_\xi'$. Then Eq.(4) becomes a fourth-order ordinary differential equation:

$$\begin{cases} X_\xi' = Y, \\ \beta_1 D Y_\xi' = Y - F[X]. \end{cases} \tag{5}$$

To make it clear under what conditions Eq.(5) produces Hopf bifurcation, it is necessary to analyse the eigenvalues of the matrix of the linearized equation in Eq.(5), evaluated at the steady-state point. According to the Hopf bifurcation theorem,[17,18] these eigenvalues, roughly speaking, should satisfy the following two points:

(1) There is a conjugate pair of complex roots whose real parts vary from negative to positive with the parameters.

(2) The other two eigenvalues always keep negative real parts.

Further analysis shows that these conditions, specifically, require that matrix A has a conjugate pair of complex roots and that $\dfrac{D_1-D_2}{D_1+D_2} < \dfrac{a_1+1}{a_1-1} + \sqrt{\left(\dfrac{a_1+1}{a_1-1}\right)^2 + \dfrac{4R}{S}}$ holds (where $R = -\dfrac{P}{2}$, $S = \dfrac{\sqrt{-\triangle}}{2}$).

The remaining problem is to test the conditions. For this, the following two lemmas are needed.

Lemma 1 If $40\mu + 6\varepsilon \le 5$, then there exist two constants, f_2 and f_1 ($f_2 > f_1 > 0$), which depend possibly upon μ and ε, such that

(1) $P < 0$ when $f \in (f_1, f_2)$;

(2) $f_1 > \dfrac{1}{2} + \dfrac{2\mu}{1-2\mu}$.

Proof: Through complicated operations, we find that $P = 0$ is equivalent to $\Phi(f) = 0$, where $\Phi(f) = af^3 + bf^2 + cf + d$ in which $a = 2$, $b = -3(1-\varepsilon) - 2(1-7\mu)$, $c = -2(\varepsilon^2 - 5\mu - 4\mu\varepsilon - \mu^2)$ and $d = (1+\varepsilon)(1+\mu)^2$. According to "Descartes' rule of signs" about polynomial roots, equation $\Phi(f) = 0$ has either two positive roots or no positive root. Notice that $\Phi'(f) = 3af^2 + 2bf + c$ and $b^2 - 3ac = 5(5-6\varepsilon-40\mu)+21\varepsilon^2+36\mu+184\mu^2 > 0$ as $40\mu+6\varepsilon \le 5$. Therefore, $\Phi'(f) = 0$ has two roots with different signatures. According to the "Isolate Theorem" of polynomial roots, we know that $\Phi(f) = 0$ must have and has only two positive real roots, denoted by f_1 and f_2, respectively. Without loss of generality, we assume

$f_2 > f_1 > 0$. Since $a_1 > 1$ as $f \in (f_1, f_2)$, we have $P < 0$. The first part of the lemma is thus proven.

From the given conditions, we know that $1 - 2u_+ \leq 0$ holds as $0 < f \leq \frac{1}{2} + \frac{2\mu}{1-2\mu}$. However, $1 - 2u_+ \leq 0$ gives $P > 0$. Hence $f_1 > \frac{1}{2} + \frac{2\mu}{1-2\mu}$. The proof of the lemma is then finished.

Lemma 2 Assume $40\mu + 6\varepsilon \leq 5$. If $0 < \varepsilon < \frac{1}{2}$, then there exist two constants, f_1' ($> f_1$) and f_2' ($< f_2$), such that $\triangle < 0$ as $f \in (f_1, f_1')$ or $f \in (f_2', f_2)$.

Proof: Notice that $P(f_1) = P(f_2) = 0$ from lemma 1. A simple operation yields $\triangle = P^2 - 4P + \frac{4(u_+ - 1 + \varepsilon)}{\varepsilon}$. Also notice that $u_+ < 1 - \varepsilon$ is equivalent to $f > \varepsilon\frac{1+\mu-\varepsilon}{1+\mu+\varepsilon}$. The conditions of the lemma imply $\frac{1}{2} + \frac{2\mu}{1-2\mu} > \varepsilon\frac{1+\mu-\varepsilon}{1+\mu+\varepsilon}$. Therefore, $\triangle(f_1) < 0$ and $\triangle(f_2) < 0$. According to the property of continuous functions, the conclusion of the lemma is easily obtained.

Remark 1. *$\triangle(f) < 0$ is not always held in the whole interval (f_1, f_2). For example, take $f = 1$, $\varepsilon = \frac{1}{16}$, $\mu = \frac{1}{200}$. Then $\triangle > 0$.*

Obviously, matrix A has eigenvalues $\lambda_\pm = \frac{-P \pm \sqrt{\triangle}}{2}$. Therefore, we have

Corollary 1 *Assume $40\mu + 6\varepsilon \leq 5$ and $0 < \varepsilon < \frac{1}{2}$. Then, as $f \in (f_1, f_1')$ or $f \in (f_2', f_2)$, matrix A has a conjugate pair of complex roots.*

Now we turn to the discussion of the existence of planar waves of Eq.(2). Since the solutions of Eq.(5) are just the planar wave solutions of Eq.(1), the following theorem holds from the Hopf bifurcation theorem.

Theorem 1 Assume $40\mu + 6\varepsilon \leq 5$, $0 < \varepsilon < \frac{1}{2}$, $D_1 \geq D_2$, $\frac{D_1 - D_2}{D_1 + D_2} < \frac{a_1+1}{a_1-1} + \sqrt{\left(\frac{a_1+1}{a_1-1}\right)^2 + \frac{4R}{S}}$. Then there exist two small intervals (f_1, f_1') and (f_2', f_2) such that Eq.(2) has planar wave solutions, as $f \in (f_1, f_1')$ or $f \in (f_2', f_2)$.

2.2. Non-planar waves

In this section some non-planar waves will be explicitly given by the singular perturbation theory.

For convenience, we consider only the two-dimensional case (a similar discussion can be applied to the three-dimensional case). Notice that the condition $D_1 \geq D_2$ is unnecessary in the following discussion.

Generally, wave phenomena involve a frequency ω, at which u oscillates at a fixed point, and a characteristic wave vector $\boldsymbol{k}$ (its norm is usually called the wavenumber, denoted by k) corresponding to a typical pattern length in the curve solution. We introduce a steady-state solution U^*, $U^* = \begin{pmatrix} u_+ \\ u_+ \end{pmatrix}$, which satisfies $F(U^*) = 0$. Thus it has also an amplitude parameter σ, $\sigma \propto \max \parallel U - U^* \parallel$, characteristic of the distance $X = U - U^*$ from the steady state, where $U = \begin{pmatrix} u \\ v \end{pmatrix}$. It is better to introduce dimensionless variables $t' = \omega t$, $\rho = k\boldsymbol{z}$. For convenience, we use t to replace t' below. With this,

$$\omega\partial_t X = k^2 D\Delta_\rho X + F(X). \tag{6}$$

Remarking 2. For $\omega \to 0$, we have low-frequency waves; $k \to 0$, long wavelength; $\sigma \to 0$, small amplitude waves.

In this paper, only the small amplitude wave is considered. For convenience, we introduce the polar coordinate system. Equation (6) is then rewritten as

$$\omega\partial_t X = k^2 D\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)X + F(X). \tag{7}$$

Notice that the waveform $X(r, \vartheta, t)$ is 2π-periodic in t and depends on the wave amplitude σ. Furthermore, the frequency ω and the square of the wave vector k^2 will vary with the amplitude parameter σ. By symmetry considerations, it is clear that the sign of σ cannot affect ω and k^2. Thus, we have the following expansions of X, ω and k^2:

$$X = \sum_{n=1}^{\infty} X_n\sigma^n, \quad \omega = \sum_{n=0}^{\infty} \omega_{2n}\sigma^{2n}, \quad k^2 = \sum_{n=0}^{\infty} k_{2n}\sigma^{2n}. \tag{8}$$

Substituting these into Eq.(7) and considering only the first-order term in σ, we have

$$\omega_0\partial_t X_1 = k_0^2 D\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\right)X_1 + AX_1. \tag{9}$$

By construction, X_1 is a 2π-periodic function. Using Fourier transformations, we see that Eq.(9) has a solution for real k_0 and ω_0 if $A - k_0^2 D$ has pure imaginary eigenvalues $\pm i\omega_0$ for at least one integer $m \geq 0$. In what follows, we limit ourselves to non-degenerate non-resonant cases when $A - k_0^2 D$ has a simple eigenvalue ($\omega_0 = 0$) or a pair of complex imaginary eigenvalues $\pm i\omega_0$ ($\omega_0 \neq 0$). Meanwhile, it is easy to show that

$$k_0^2 = \frac{-P}{D_1 + D_2}, \quad \omega_0 = (-b_1) - \left(\frac{D_1 + a_1 D_2}{D_1 + D_2}\right)^2. \tag{10}$$

Therefore, combined with lemma 1, the following conclusion is easily attained.

Theorem 2 If $40\mu+6\varepsilon<5$ and $\frac{f}{\varepsilon}\left(\frac{u_+-\mu}{u_++\mu}\right)\geq\left(\frac{D_1+a_1D_2}{D_1+D_2}\right)^2$, there exist f_2 and f_1 ($f_2>f_1>0$), such that Eq.(9) has nontrivial solutions as $f\in(f_1,\,f_2)$.

Consider the case $\omega_0\neq 0$. According to the nature of the Bessel function,[19] Eq.(9) has *spiral wave* solutions of the form

$$X_1^{(m)}=X_1(\boldsymbol{r},\omega_0t+m\vartheta)\equiv J_m(r)\mathrm{Re}\left(\xi_+\mathrm{e}^{i(\omega_0t+m\vartheta)}\right),\tag{11}$$

where $J_m(r)$ is an mth-order Bessel function, $m=\pm1,\pm2,\cdots$, $i=\sqrt{-1}$, and ξ_+ is an eigenvector corresponding to the eigenvalue $+i\omega_0$ of $A-k_0^2D$. It is easy to show that $\xi_+=\begin{pmatrix}1\\1+k_0^2D_2+i\omega_0\end{pmatrix}$. The patterns of these waves have fixed nodes at the zeros of $J_m(r)$ and rotate at an angular speed, $\frac{\omega_0}{m}$ ($m=\pm1,\,\pm2,\,\cdots$).

The first-order theory of a non-planar wave with respect to the wave amplitude can predict that for the wave vector k_0 a spiral wave solution (of strictly zero amplitude) exists with a frequency ω_0. Nonzero-amplitude wave phenomena must only be at $|k^2-k_0^2|>0$. When these small-amplitude solutions exist, they represent a balance between the linearized dynamics of perturbations with a wave vector k and nonlinearity.

In addition, Eq.(9) has the solutions of the form:

$$X_1^{(m)}=X_1(\boldsymbol{r},\omega_0t+m\vartheta)\equiv J_m(r)\sin(m\vartheta)\mathrm{Re}\left(\xi_+\mathrm{e}^{it}\right).\tag{12}$$

These solutions are sometimes called standing waves.

Notice that when $m\vartheta=\pi$ ($m=\pm1,\,\pm2,\,\cdots$), $X_1^{(m)}=0$, and the zeros of $J_m(r)$ also make $X_1^{(m)}$ null. All these zeros consist of the nodes of the standing waves.

Finally, by introducing the second-class Bessel function Y_m:$Y_m=\lim\limits_{\mu\to0}Y_\mu(r)$ (where $Y_\mu(r)=\frac{1}{\sin(\mu\pi)}[J_\mu(r)\cos(\mu\pi)-J_{-\mu}(r)]$, and μ is non-integral), and taking the linear combination, $H_m^\pm=J_m\pm Y_m$, ($m=\pm1\ \pm2\ \cdots$), then Eq.(9) has also the solutions of the form

$$X_1^{(m)}=X_1(\boldsymbol{r},\omega_0t+m\vartheta)\equiv\mathrm{Re}\left(\xi_+H_m^\pm(r)\mathrm{e}^{i(t+m\vartheta)}\right),\tag{13}$$

where $m=\pm1,\ \pm2,\cdots$. The waves of this form display the radially propagating phenomena.

Next, we consider the asymptotic behaviours of Eq.(13) with respect to r. From the nature of the Bessel function, it follows from Eq.(13)

$$X_1^{(m)}\sim\sqrt{\frac{2r}{\pi}}\mathrm{Re}\left(\xi_+\mathrm{e}^{i\Phi^\pm}\right),\tag{14}$$

as $r\to+\infty$, where $\Phi^\pm=\pm\left(r-\frac{1}{2}m\pi-\frac{1}{4}\pi\right)+t+m\vartheta$, $m=0,\,\pm1,\,\pm2,\,\cdots$. Therefore, when r is sufficiently large, such a wave propagates approximately along the radial direction. However, $H_m^\pm\sim r^{-m}$ as $r\to0$ (when $m=0$, $H_0^\pm\sim\log r$ for $r\to0$); in this case the above expansions on the small amplitude σ are invalid because they diverge as $r\to0$. The region of non-uniformity takes place near the central core $r=0$.

3. Stability of waves

Generally, it is difficult to give a criterion to judge the stability of waves in the Oregonator model, but is possible for small-amplitude planar waves. In this section, a sufficient condition to ensure the stability of planar waves is given.

We assume $D_1>0$ and $D_2>0$. Let $Y^*(z,\,t)=X(\omega t-\boldsymbol{kz})$ be a planar wave solution of Eq.(2). To investigate the stability of the wave, we introduce a small perturbation in the neighbourhood of the wave and consider the following linearized perturbation equation

$$W_t'=B(Y^*(z,\,t))W+D\Delta W,\tag{15}$$

where $W=X-Y^*(z,\,t)$ and matrix B is defined as

$$B=(b_{ij})=((\partial F_i/\partial X_j)(X))=\begin{pmatrix}\frac{1}{\varepsilon}\left(1-2u-\frac{2f\mu v}{(u+\mu)^2}\right) & -\frac{f(u-\mu)}{\varepsilon(u+\mu)}\\ 1 & -1\end{pmatrix},$$

in which u and v are functions with $\omega t-\boldsymbol{kz}$ as the argument.

For research convenience, we now change the variables in Eq.(15) to those in a coordinate system moving along the wave by setting $\tau=t$ and $\xi=\boldsymbol{nz}-\frac{\omega}{k}t$ (where $\boldsymbol{n}$ is a unit constant vector). With this, Eq.(15) becomes

$$W_\tau'=\frac{\omega}{k}W_\xi'+B(X(-k\xi))W+DW_{\xi\xi}''.\tag{16}$$

We want to know if there are perturbations bounded for all z, and also for all ξ, which grow in

time t (or τ). This will be the case if the ordinary differential function of ξ has some points of its spectrum in the right half-plane, i.e., if Eq.(16) has the solution of the form $W = e^{h\tau}\Gamma(\xi)$ with the bounded Γ and $\mathrm{Re}(h) > 0$. This gives an eigenvalue problem of the ordinary differential equation

$$h\Gamma = \frac{\omega}{k}\Gamma'_\xi + B(X(-k\xi)) + D\Gamma''_{\xi\xi}. \tag{17}$$

Let $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ (where $\eta_1 = \Gamma$ and $\eta_2 = \Gamma'_\xi$) and $C = \begin{pmatrix} 0 & 1 \\ D^{-1}(hI - B(X(-k\xi))) & \frac{\omega}{k}D^{-1} \end{pmatrix}$ (where I is a second-order unit matrix). Equation (17) can be rewritten as

$$\eta'_\xi = C\eta. \tag{18}$$

Since X is a 2π-periodic function with respect to ξ, so is the matrix C. Theoretically, we may show the stability of solutions of Eq.(18) by determining the signature of their Floquet exponents. But this does not seem to be feasible since we do not know the explicit expression of the planar wave solution. Because of this, the following discussion is restricted to the small-amplitude planar wave. Thus, $B(X(-k\xi)) \sim \begin{pmatrix} a_1 & b_1 \\ 1 & -1 \end{pmatrix} = A$ and Eq.(18) has certain solutions of the form $\Gamma = \begin{pmatrix} a \\ b \end{pmatrix} \mathrm{e}^{ie\xi}$, where a and b are (possibly complex) constants and e is real. Substituting Γ into Eq.(18) and considering the nontrivial solution yield

$$\begin{aligned} &D_1D_2e^4 - (D_1+D_2)\beta ie^3 \\ &+ \left[(1+h)D_1 + (h-a_1)D_2 - \beta^2\right]e^2 \\ &- (1+2h-a_1)\beta ie - (a_1-h)(1+h) - b_1 = 0, \end{aligned} \tag{19}$$

where $\beta = \dfrac{\omega}{k}$. Setting $h = h_1 + ih_2$ and balancing the real and imaginary parts of Eq.(19), we can obtain the following expressions:

$$\begin{aligned} &e^4 + \frac{(1+h_1)D_1 + (h_1-a_1)D_2 - \beta^2}{D_1+D_2} \\ &e^2 + \frac{2\beta h_2}{D_1D_2}e + \frac{h_1^2 - h_2^2 + Ph_1 + Q}{D_1D_2} = 0, \end{aligned} \tag{20}$$

$$\begin{aligned} &-(D_1+D_2)\beta e^3 + h_2(D_1+D_2)e^2 \\ &-(P+2h_1)\beta e + (2h_1+P)h_2 = 0. \end{aligned} \tag{21}$$

Equation (21) can be further turned into

$$(h_2 - \beta e)\left[(D_1+D_2)e^2 + 2h_1 + P\right] = 0. \tag{22}$$

If $h_2 - \beta e = 0$, then no matter what value h_2 takes, no rigorously positive h_1 can be found from Eq.(20). Therefore, Eq.(21) yields

$$e = \pm\sqrt{\frac{-(2h_1+P)}{D_1+D_2}}. \tag{23}$$

For convenience, we consider the equal diffusion case, $D_1 = D_2$, and we denote the common value by D_0. Substituting Eq.(23) into Eq.(20) yields

$$-\frac{\triangle}{4} - h_2^2 + \frac{2h_1+P}{2D_0}\beta^2 \pm 2\beta h_2\sqrt{\frac{-(2h_1+P)}{D_1+D_2}} = 0, \tag{24}$$

which further becomes

$$\begin{aligned} &\frac{1}{D_0}\beta^4h_1^2 + 2\beta^2\left[-\frac{\triangle}{4} + h_2^2 + \frac{P}{4D_0}\beta^2\right]h_1 \\ &+ \left(-\frac{\triangle}{4} - h_2^2 + \frac{P}{4D_0}\beta^2\right)^2 + 2P\beta^2h_2^2 = 0. \end{aligned} \tag{25}$$

In order to guarantee that h_1 is positive, h_2 must satisfy

$$\begin{cases} -\frac{\triangle}{4} + h_2^2 + \frac{P}{4D_0}\beta^2 < 0, \\ \left(-\frac{\triangle}{4} - h_2^2 + \frac{P}{4D_0}\beta^2\right)^2 + 2P\beta^2h_2^2 > 0 \end{cases}. \tag{26}$$

Because of this, the conclusion immediately follows.

Theorem 3 Under the conditions of theorem 1, there exist two small intervals $(f_1,\ f'_1)$ and $(f'_2,\ f_2)$ such that the small-amplitude planar wave solutions of Eq.(1) are linearly stable if $f \in (f_1,\ f'_1)$ or $f \in (f'_2,\ f_2)$ and $0 < \beta < \sqrt{\dfrac{\triangle}{P}D_0}$.

As a by-product, the above analysis yields an appraisal for h_2:

$$0 < h_2^2 < \frac{-(D_0\triangle + 3P^2) - |\beta|\sqrt{P(D_0\triangle + P\beta^2)}}{2D_0}. \tag{27}$$

4. Conclusion

We have rigorously proven the existence of planar waves in the excitable media of the Oregonator model. By expanding wave pattern solutions into the series of Bessel functions with respect to the small wave amplitude and by analysing the leading terms in the expansion, some non-planar waves in the model are found for the first time. For small-amplitude planar waves, some sufficient conditions for their linear stability are also given. Further study for the stability of the non-planar waves is still necessary.

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