Vol 11 No 12, December 2002
1009-1963/2002/11(12)/1228-06
Chinese Physics
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A connection theory for a nonlinear differential constrained system*

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(Received 14 May 2002; revised manuscript received 16 August 2002)

An Ehresmann connection on a constrained state bundle defined by nonlinear differential constraints is constructed for nonlinear nonholonomic systems. A set of differential constraints is integrable if and only if the curvature of the Ehresmann connection vanishes. Based on a geometric interpretation of d–δ commutation relations in constrained dynamics given in this paper, the complete integrability conditions for the differential constraints are proven to be equivalent to the three requirements upon the conditional variation in mechanics: (1) the variations belong to the constrained manifold; (2) the time derivative commutes with variational operator; (3) the variations satisfy the Chetaev's conditions.

Keywords: nonholonomic constraint, Ehresmann connection, integrability condition, d–δ commutation relation

PACC: 0320, 0240

1. Introduction

Nonholonomic constraints are often encountered in the studies of nonholonomic mechanics, field theories and motion planning.[1−10] A nonholonomic constraint does not affect the configuration manifold but the fibres only, which corresponds to the restriction upon the freedom of mechanical systems. So the constrained submanifold does not have a symplectic (or presymplectic) structure of tangent bundle (or contact manifold). Therefore, in terms of calculus of variations in classical nonholonomic mechanics, it is impossible for nonholonomic constraints to satisfy the following three requirements simultaneously: (1) the variations belong to constrained submanifold; (2) the variational operator commutes with the time derivative; (3) the variations obey the Chetaev's conditions, unless the differential constraints are integrable in the sense of Frobenius theorem.[11]

If a mechanical system is subject to linear nonholonomic constraints, it can be proven that there exists a fibre bundle structure on a constrained submanifold, whose horizontal distribution, called the Ehresmann connection, is determined by the nonholonomic constraints. The non-vanishing of curvature form of the Ehresmann connection corresponds to the non-integrability of the constraints, which was proven in Ref.[12] to be equivalent to the condition that Suslov's variation commutes with the time derivative.

In this paper, we generalize the result to the nonlinear nonholonomic constrained systems which may be important for nonholonomic motion planning. In section 2 we construct an Ehresmann connection on a constrained submanifold of 1-jet bundle J_1M. The curvature forms of the connections are derived in section 3. It is proven that a set of differential constraints is integrable if and only if the corresponding curvature form vanishes. In section 4, the connection theory is applied to analyse d–δ commutation relations in calculus of variations. We discover that these relations correspond to the contraction of the curvature form with a dynamical vector field of mechanical system and its dual basic 1-form relating to the "irreducible coordinates".[13] Considering this geometrical explanation of d–δ commutation relations, it is proven that the above three requirements upon the variational operator can be taken as a criterion of integrability for differential constraints. Finally, an illustrative ex-

*Project supported by the National Natural Science Foundation of China (Grant No 10175032), the Natural Science Foundation of Liaoning Province, China (Grant Nos 002083 and 2001101024), and the Science Research Foundation of Liaoning Education Bureau, China (Grant Nos 990111004 and 20021004).

ample is given. The Einstein's summation convention is used throughout this paper: $s = 1, \ldots, n$; μ, $\sigma = 1, 2, \ldots, \varepsilon$; $\beta = 1, 2, \ldots, g = n - \varepsilon$, where ε denotes the degrees of freedom of the mechanical system.

2. Ehresmann connection on a constrained state bundle

An Ehresmann connection on a tangent bundle or 1-jet bundle is usually defined by the Lie derivative of vertical endomorphism with respect to the dynamical vector field, which determines the horizontal distributions on the tangent bundle.[14−16] By Ehresmann connection here we mean a horizontal distribution on the constrained fibre bundle defined by the constraint equations.

Consider a time-dependent mechanical system with n generalized coordinates. Let M be its smooth extended configuration manifold of dimension $n+1$ with local coordinates $\{t, q^s\}$ which is a fibred manifold over R with canonical projection $\pi : M \to R$. TM denotes the tangent bundle to M and J_1M the 1-jet bundle. Suppose that the system is subject to g first-order nonlinear nonholonomic constraints:

$$\dot{q}^{\varepsilon+\beta} = \varphi^{\beta}\left(t, q^{s}, \dot{q}^{\sigma}\right). \tag{1}$$

Problems with nonlinear constraints occur frequently in two fields: optimal control and economical dynamics.[15,17] This construction distinguishes two groups of coordinates, q^{σ} and $q^{\varepsilon+\beta}$. If the coordinate transformations on M keep this distinction, the configuration manifold M is of fibration structure.[18]

Let VM denotes the vertical subspace of TM with respect to the projection $\tau : M \to M_0$, where $M_0 \subset M$ is a manifold of dimension $\varepsilon + 1$. $J_1M_0 \subset TM_0$ is a 1-jet manifold with respect to $\pi_0 : M_0 \to R$ and $\tau^*(J_1M_0) = M \times_{M_0} J_1M_0$ the pull-back bundle over M.

Definition 1 Let $i : \tau^*(J_1M_0) \to J_1M$ be an imbedding defined by the constraint equations (1), then $N = i(\tau^*(J_1M_0))$ is called a constrained state submanifold of J_1M, and the bundle $\rho : N \to J_1M_0$ is referred to as constrained state bundle.

Let VN be the vertical subspace of TN with respect to ρ and $\rho^*(T(J_1M_0))$=$N \times_{J_1M_0} T(J_1M_0)$ the pull-back bundle over N. $\chi(\rho)$ denotes the set of vector fields along ρ, i.e. the set of sections $S : N \to \rho^*(T(J_1M_0))$. Then the following exact sequence

$$0 \to VN \to TN \to \rho^*\left(T\left(J_1M_0\right)\right) \to 0 \tag{2}$$

determines an Ehresmann connection on the bundle $\rho : N \to J_1M_0$.

Let $H : \chi(\rho) \to TN$, then $\mathrm{d}\rho \cdot H = \mathrm{id}_{\chi(\rho)}$. In local coordinates,

$$H_0 = H\left(\partial/\partial t\right) = \partial/\partial t + \overline{\Gamma}_0^{\varepsilon+\beta}\partial/\partial q^{\varepsilon+\beta}, \tag{3a}$$

$$H_{\sigma} = H\left(\partial/\partial q^{\sigma}\right) = \partial/\partial q^{\sigma} + \overline{\Gamma}_{\sigma}^{\varepsilon+\beta}\partial/\partial q^{\varepsilon+\beta}, \tag{3b}$$

$$H_{\dot{\sigma}} = H\left(\partial/\partial \dot{q}^{\sigma}\right) = \partial/\partial \dot{q}^{\sigma} + \overline{\Gamma}_{\dot{\sigma}}^{\varepsilon+\beta}\partial/\partial q^{\varepsilon+\beta}, \tag{3c}$$

where $\overline{\Gamma}_0^{\varepsilon+\beta}$, $\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}$, $\overline{\Gamma}_{\dot{\sigma}}^{\varepsilon+\beta} \in C^{\infty}(N)$, and $\{H_0, H_{\sigma}, H_{\dot{\sigma}}\}$ expands the horizontal distribution on N. The canonical projection ρ induces a mapping

$$H' : N \to J_1\rho,$$

$$\left(t,\ q^{\sigma},\ \dot{q}^{\sigma},\ q^{\varepsilon+\beta}\right) \to$$

$$\left(t,\ q^{\sigma},\ \dot{q}^{\sigma},\ q^{\varepsilon+\beta},\ q_0^{\varepsilon+\beta},\ q_{\sigma}^{\varepsilon+\beta},\ q_{\dot{\sigma}}^{\varepsilon+\beta}\right),$$

with

$$\begin{aligned}
\overline{\Gamma}_0^{\varepsilon+\beta} &= q_0^{\varepsilon+\beta} = \varphi^{\beta} - \dot{q}^{\sigma}\partial\varphi^{\beta}/\partial\ \dot{q}^{\sigma}, \\
\overline{\Gamma}_{\sigma}^{\varepsilon+\beta} &= q_{\sigma}^{\varepsilon+\beta} = \partial\varphi^{\beta}/\partial\dot{q}^{\sigma}, \\
\overline{\Gamma}_{\dot{\sigma}}^{\varepsilon+\beta} &= q_{\dot{\sigma}}^{\varepsilon+\beta} = 0,
\end{aligned} \tag{4}$$

where $J_1\rho$ is 1-jet bundle of $\rho : N \to J_1M_0$.

The bases of vector fields on N can be taken as

$$H_0,\ H_{\sigma},\ H_{\dot{\sigma}} = \partial/\partial\dot{q}^{\sigma}, \qquad V_{\beta} = \partial/\partial q^{\varepsilon+\beta}, \tag{5}$$

with dual basic 1-forms

$$\mathrm{d}t,\quad \mathrm{d}q^{\sigma},\quad \mathrm{d}\dot{q}^{\sigma},\quad \xi^{\varepsilon+\beta} = \mathrm{d}q^{\varepsilon+\beta} - \overline{\Gamma}_{\sigma}^{\varepsilon+\beta}\mathrm{d}q^{\sigma} - \overline{\Gamma}_0^{\varepsilon+\beta}\mathrm{d}t. \tag{6}$$

With the help of these bases, the horizontal and vertical projections of vector fields on N can be represented as

$$\begin{aligned}
P_{\mathrm{H}} &= H_0 \otimes \mathrm{d}t + H_{\sigma} \otimes \mathrm{d}q^{\sigma} + H_{\dot{\sigma}} \otimes \mathrm{d}\dot{q}^{\sigma}, \\
P_V &= \partial/\partial q^{\varepsilon+\beta} \otimes \xi^{\varepsilon+\beta}.
\end{aligned} \tag{7}$$

Obviously,

$$P_{\mathrm{H}}(TN) = H\left(\rho^*\left(T\left(J_0M\right)\right)\right) \triangleq H_{\rho},$$

where H_{ρ} denotes the horizontal distribution on N.

3. The curvature form of the connection and integrability conditions

By the definition of the fibre bundle, $\rho : N \to J_1M_0$ is locally diffeomorphic to trivial bundles. So the connections defined above are not flat in general,

unless their curvature forms vanish. There exist several methods to determine the curvature of a connection, e.g., via the Nijenhuis tensor of a horizontal or vertical projection operator,[14,19,20] the commutator of horizontal exterior differentiations,[21] and the exterior covariant differentiation of a vector-valued connection 1-form on the bundle,[22,23] etc. In this paper, the last method is utilized to define the curvature forms of the connections.

The above Ehresmann connection H on the bundle $\rho : N \to J_1M_0$ is defined by the splitting of an exact sequence 2. Following the connection theories of principal fibre bundles, these connections can also be defined equivalently by a vector-valued 1-form.

Definition 2 A VN-valued connection 1-form $\omega_{\rm H}$ on N is the projection $P_V : TN \to VN$, i.e.

$$\omega_{\rm H} = P_{\rm V} = \partial/\partial q^{\varepsilon+\beta} \otimes \xi^{\varepsilon+\beta} \tag{8}$$

and its curvature form is a VN-valued 2-form R

$$R = \nabla P_{\rm V} = {\rm d}P_{\rm V} \cdot P_{\rm H}. \tag{9}$$

Substituting Eq.(7) into Eq.(9), we have, in local coordinates,

$$\begin{aligned} R = & \Big\{ [H_\sigma(\overline{\Gamma}_0^{\varepsilon+\beta}) - H_0(\overline{\Gamma}_\sigma^{\varepsilon+\beta})]{\rm d}t \wedge {\rm d}q^\sigma \\ & + \frac{1}{2}[H_\mu(\overline{\Gamma}_\sigma^{\varepsilon+\beta}) - H_\sigma(\overline{\Gamma}_\mu^{\varepsilon+\beta})]{\rm d}q^\sigma \wedge {\rm d}q^\mu \\ & + \frac{\partial \overline{\Gamma}_0^{\varepsilon+\beta}}{\partial \dot{q}^\sigma}{\rm d}t \wedge {\rm d}\dot{q}^\sigma \\ & + \frac{\partial \overline{\Gamma}_\sigma^{\varepsilon+\beta}}{\partial \dot{q}^\mu}{\rm d}q^\sigma \wedge {\rm d}\dot{q}^\mu \Big\} \otimes \partial/\partial q^{\varepsilon+\beta}. \end{aligned} \tag{10}$$

The conditions of complete integrability for general constraint equations (1) can also be geometrically characterized via the curvature form of the connection.

Proposition 3 Let R be the curvature form of connection H defined by a set of nonlinear differential constraints, then the necessary and sufficient condition for the constraint equations to be integrable is that the curvature form R vanishes.

The proof of this proposition is very similar to the case of linear constraints[12] and can be neglected. It follows from Eq.(10) that $R = 0$ if and only if

$$\begin{aligned} & H_0\left(\overline{\Gamma}_\sigma^{\varepsilon+\beta}\right) - H_\sigma\left(\overline{\Gamma}_0^{\varepsilon+\beta}\right) = 0, \\ & H_\sigma\left(\overline{\Gamma}_\mu^{\varepsilon+\beta}\right) - H_\mu\left(\overline{\Gamma}_\sigma^{\varepsilon+\beta}\right) = 0, \\ & \frac{\partial \overline{\Gamma}_0^{\varepsilon+\beta}}{\partial \dot{q}^\sigma} = 0, \qquad \frac{\partial \overline{\Gamma}_\sigma^{\varepsilon+\beta}}{\partial \dot{q}^\mu} = 0. \end{aligned}$$

Further examination in the next section shows that these integrability conditions make the nonlinear differential constraints reduce to linear ones. Since the basis $\{H_0, H_\sigma, H_{\dot{\sigma}}\}$ of horizontal distribution H_ρ satisfies the following commutation relations

$$\begin{aligned} & [H_0,\ H_0] = [H_{\dot{\sigma}},\ H_{\dot{\mu}}] = 0, \\ & [H_0,\ H_\sigma] = \left[H_0\left(\overline{\Gamma}_\sigma^{\varepsilon+\beta}\right) - H_\sigma\left(\overline{\Gamma}_0^{\varepsilon+\beta}\right)\right]\partial/\partial q^{\varepsilon+\beta}, \\ & [H_\sigma,\ H_\mu] = \left[H_\sigma\left(\overline{\Gamma}_\mu^{\varepsilon+\beta}\right) - H_\mu\left(\overline{\Gamma}_\sigma^{\varepsilon+\beta}\right)\right]\partial/\partial q^{\varepsilon+\beta}, \\ & [H_0,\ H_{\dot{\sigma}}] = -\frac{\partial \overline{\Gamma}_0^{\varepsilon+\beta}}{\partial \dot{q}^\sigma}\partial/\partial q^{\varepsilon+\beta}, \\ & [H_\sigma,\ H_{\dot{\mu}}] = -\frac{\partial \overline{\Gamma}_\sigma^{\varepsilon+\beta}}{\partial \dot{q}^\mu}\partial/\partial q^{\varepsilon+\beta}, \end{aligned}$$

the condition $R = 0$ means that

$$\begin{aligned} & [H_0,\ H_0] = [H_{\dot{\sigma}},\ H_{\dot{\mu}}] = [H_0,\ H_\sigma] \\ & = [H_\sigma,\ H_\mu] = [H_0,\ H_{\dot{\sigma}}] = [H_\sigma,\ H_{\dot{\mu}}] = 0, \end{aligned}$$

which implies that the distribution H_ρ is involutive. From the Cartan's structure equation, the non-involution of horizontal distribution on N can be characterized by the relation

$$R\,(X,\ Y) = -P_{\rm V}\,([X,\ Y]), \quad \forall\ X,\ Y \in H_\rho. \tag{11}$$

Now we can turn to the applications in calculus of variations.

4. Applications to calculus of variations

Noether's symmetries are essentially different from the dynamical mechanics, physics and control theory, etc. If the differential constraints are present, some requirements must be exerted to variations and we obtain the so-called conditional variations. Consider a mechanical system subject to a set of differential constraints of the form (1), then the variations, in general, do not satisfy the following three requirements simultaneously:

$$\frac{\rm d}{{\rm d}t}\delta q^s = \delta \dot{q}^s, \tag{12a}$$

$$\delta \dot{q}^{\varepsilon+\beta} = \frac{\partial \varphi^{\varepsilon+\beta}}{\partial \dot{q}^\sigma}\delta \dot{q}^\sigma + \frac{\partial \varphi^{\varepsilon+\beta}}{\partial q^s}\delta q^s \tag{12b}$$

$$\delta q^{\varepsilon+\beta} = \frac{\partial \varphi^{\varepsilon+\beta}}{\partial \dot{q}^\sigma}\delta q^\sigma, \tag{12c}$$

unless the constraints are integrable. In Chetaev's model of nonholonomic mechanics, the conditions

(12b) and (12c) are selected to define the Suslov's variation that keeps the variations of orbits belonging to the constrained submanifold and meets Chetaev's conditions. But, in this case the variational operator δ does not commute with the differential operator d. Usually their commutation relations are called d–δ commutation relations. Nevertheless, they have never been well understood in the study of classical constrained dynamics. In the framework established above, we can characterize the d–δ commutation relations geometrically and prove that the above three conditions can be accepted as a criterion of integrability for differential constraint equations.

4.1. The geometrical meaning of d–δ commutation relations

In order to make clear the significance of the curvature 2-form R in calculus of variations, we construct another tetrad basis of vector fields, or moving frame, on N. To this end, we first introduce a dynamical vector field on the constrained manifold.

Definition 4 A dynamical vector field $\boldsymbol{Z}$ on the constrained submanifold N is a vector field satisfying the condition:

$$i_{\vec{Z}}\mathrm{d}t = 1, \quad i_{\vec{Z}}\theta^{\sigma} = 0, \quad i_{\vec{Z}}\xi^{\varepsilon+\beta} = 0,$$

where θ^{σ} is Cartan's contact 1-form. In local coordinates,

$$\begin{aligned}\boldsymbol{Z} =& \partial/\partial t + \dot{q}^{\sigma}\partial/\partial q^{\sigma} + \varphi^{\beta}(t, q^{s}, \dot{q}^{\mu})\,\partial/\partial q^{\varepsilon+\beta} \\ &+ f^{\sigma}(t, q^{s}, \dot{q}^{\mu})\,\partial/\partial\dot{q}^{\sigma}.\end{aligned}$$

Substituting $\boldsymbol{Z}$ for H_0 in the old tetrad (5), we obtain a new tetrad basis:

$$Z, \quad H_{\sigma}, \quad \partial/\partial\dot{q}^{\sigma}, \quad \partial/\partial q^{\varepsilon+\beta}.$$

The dual basic 1-forms correspondingly change into

$$\mathrm{d}t, \quad \theta^{\sigma}, \quad \omega^{\sigma}, \quad \xi^{\varepsilon+\beta},$$

with $\omega^{\sigma} = \mathrm{d}\dot{q}^{\sigma} - f^{\sigma}\mathrm{d}t$. By means of this tetrad, the curvature form R (see equation (10)) can be transformed into

$$\begin{aligned}R =& \Big\{ -T_{\sigma}^{\varepsilon+\beta}\mathrm{d}t\wedge\theta^{\sigma} + \frac{\partial\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}}{\partial\dot{q}^{\mu}}\theta^{\sigma}\wedge\omega^{\mu} + \frac{1}{2}[H_{\mu}(\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}) \\ &- H_{\sigma}(\overline{\Gamma}_{\mu}^{\varepsilon+\beta})]\theta^{\sigma}\wedge\theta^{\mu}\Big\}\otimes\partial/\partial q^{\varepsilon+\beta}, \qquad (13)\end{aligned}$$

where

$$T_{\sigma}^{\varepsilon+\beta} = \boldsymbol{Z}\left(\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}\right) - H_{\sigma}\left(\varphi^{\beta}\right). \qquad (14)$$

Because of the duality of the new tetrad and the above basic 1-forms, we have the contraction of R with Z and $\xi^{\varepsilon+\beta}$

$$R\left(Z,\ \xi^{\varepsilon+\beta}\right) = -T_{\sigma}^{\varepsilon+\beta}\theta^{\sigma}, \qquad (15)$$

where Z and $\xi^{\varepsilon+\beta}$, in calculus of variations, correspond to $\mathrm{d}/\mathrm{d}t$ and differentials of quasi-coordinates, respectively. This equation reminds us of the so-called d–δ commutation relations. Substituting Eqs.(3a)–(3c) and (4) into Eq.(14), then

$$T_{\sigma}^{\varepsilon+\beta} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi^{\beta}}{\partial\dot{q}^{\sigma}} - \frac{\partial\varphi^{\beta}}{\partial q^{\sigma}} - \frac{\partial\varphi^{\beta}}{\partial\dot{q}^{\varepsilon+\gamma}}\frac{\partial\varphi^{\gamma}}{\partial\dot{q}^{\sigma}},$$

which is, in calculus of variations, just the commutator of time derivative operator $\mathrm{d}/\mathrm{d}t$ and variational operator δ, i.e. it satisfies

$$\left[\delta,\ \frac{\mathrm{d}}{\mathrm{d}t}\right] q^{\varepsilon+\beta} = -T_{\sigma}^{\varepsilon+\beta}\delta q^{\sigma},$$

where $[\,.\ ,\ .\,]$ denotes the usual Lie bracket of the two operators. Thus we have proven an interesting result:

Proposition 5 H is an Ehresmann connection on the constrained fibre bundle $\rho : N \to J_1M_0$ defined by the differential constraint equations (1), and R is its curvature form. Z denotes the dynamical vector field of a mechanical system and $\xi^{\varepsilon+\beta}$ its dual 1-form corresponding to the irreducible coordinates. Then the contraction of R with Z and $\xi^{\varepsilon+\beta}$ is just the geometrical formulation of d–δ commutation relations in classical calculus of variations.

This proposition gives the geometrical meaning of classical d–δ commutation relations and shows that they are covariant under the coordinate transformations on the constrained manifold.

4.2. Integrability conditions in terms of variational operator

Adopting the new tetrad bases in this section, we can obtain new integrability conditions in local coordinates.

Proposition 6 The curvature R vanishes if and only if $T_{\sigma}^{\varepsilon+\beta} = 0$.

Proof According to Eq.(13), $T_{\sigma}^{\varepsilon+\beta} = 0$ if $R = 0$. Let

$$T_{\sigma}^{\varepsilon+\beta} = \ddot{q}^{\mu}\frac{\partial\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}}{\partial\dot{q}^{\mu}} + T\left(\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}\right) - H_{\sigma}\left(\varphi^{\beta}\right) = 0,$$

then the independence of $\ddot{q}$ leads to

$$\frac{\partial\overline{\Gamma}_{\mu}^{\varepsilon+\beta}}{\partial\dot{q}^{\sigma}} = 0, \qquad T\left(\overline{\Gamma}_{\sigma}^{\varepsilon+\beta}\right) - H_{\sigma}\left(\varphi^{\beta}\right) = 0,$$

which means

$$\overline{\Gamma}{}^{\varepsilon+\beta}_{\sigma} = \frac{\partial \varphi^{\beta}}{\partial \dot{q}^{\sigma}} = F^{\varepsilon+\beta}_{\sigma}\left(t,\ q^{s}\right)$$

or

$$\varphi^{\beta} = F^{\varepsilon+\beta}_{\sigma}\left(t,\ q^{s}\right)\dot{q}^{\sigma} + F^{\varepsilon+\beta}\left(t,\ q^{s}\right).$$

It follows that

$$H_{\mu} = \overline{h}_{\mu} = \partial/\partial q^{\mu} + F^{\varepsilon+\alpha}_{\mu}\partial/\partial q^{\varepsilon+\alpha}$$

$$\frac{1}{2}\left[H_{\mu}\left(\overline{\Gamma}{}^{\varepsilon+\beta}_{\sigma}\right) - H_{\sigma}\left(\overline{\Gamma}{}^{\varepsilon+\beta}_{\mu}\right)\right] = \frac{1}{2}\overline{C}{}^{\beta}_{\mu\sigma}$$

$$T\left(\overline{\Gamma}{}^{\varepsilon+\beta}_{\mu}\right) - H_{\mu}\left(\varphi^{\beta}\right) = \overline{t}{}^{\varepsilon+\beta}_{\mu} = 0,$$

where $\overline{C}{}^{\beta}_{\mu\sigma}$ and $\overline{t}{}^{\varepsilon+\beta}_{\mu}$ are formally the same as $C^{\beta}_{\mu\sigma}$ and $t^{\varepsilon+\beta}_{\mu}$ in Ref.[12], respectively. Then we have

$$\overline{C}{}^{\beta}_{\mu\sigma} = \overline{D}{}^{\beta}_{0\sigma} = 0.$$

Substituting the above equations into Eq.(13), then $R = 0$.

It is evident that, if $T^{\varepsilon+\beta}_{\sigma} = 0$, the nonlinear constraints reduce to the linear integrable constraints, and vice versa. Combining this proposition and proposition (3), it is easy to achieve:

Corollary 7 The nonlinear differential constraints (1) are integrable if and only if $T^{\varepsilon+\beta}_{\sigma} = 0$.

On the basis of this corollary and the geometrical explanation on d–δ commutation relations, we immediately obtain

Proposition 8 The constraint equations are integrable in Frobenius sense if and only if the Suslov's variation commutes with differentiation, i.e. there exists a variational operator δ satisfying the three conditions (12a)–(12c) simultaneously.

Thus we have verified the equivalence between the integrability of differential constraints and the three requirements (12a)–(12c) upon variational operator.

Appell-Hamel's example. Consider a unit mass moving in a uniform gravitational field g. Let the local coordinates of the configuration manifold M be $x = q^1$, $y = q^2$, $z = q^3$. Then the dynamical function is $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - gz$. Suppose the particle is subject to an first-order nonholonomic constraint

$$\dot{z} = a\sqrt{\dot{x}^2 + \dot{y}^2}.$$

Then the coordinates of the constraint manifold N take $\{x, y, \dot{x}, \dot{y}\}$ and the dynamical function embedded in the constraint is $L' = (1+a^2)(\dot{x}^2 + \dot{y}^2)/2 - gz$. Obviously, this system is not a Chaplygin's system. We can find the dynamical vector field to be

$$\begin{aligned}\boldsymbol{Z} =& \dot{x}\partial/\partial x + \dot{y}\partial/\partial y + a\sqrt{\dot{x}^2+\dot{y}^2}\partial/\partial z \\ &+ \frac{ag\dot{x}}{(1+a^2)\sqrt{\dot{x}^2+\dot{y}^2}}\partial/\partial\dot{x} \\ &+ \frac{ag\dot{y}}{(1+a^2)\sqrt{\dot{x}^2+\dot{y}^2}}\partial/\partial\dot{y}.\end{aligned}$$

The connection coefficients are

$$\overline{\Gamma}{}^{3}_{0} = 0, \quad \overline{\Gamma}{}^{3}_{1} = \frac{a\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \quad \overline{\Gamma}{}^{3}_{2} = \frac{a\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}.$$

The horizontal distributions on N are expanded by

$$H_0 = \partial/\partial t, \quad H_1 = \partial/\partial q^1 + \overline{\Gamma}{}^{3}_{1}\partial/\partial z,$$

$$H_2 = \partial/\partial q^2 + \overline{\Gamma}{}^{3}_{2}\partial/\partial z,$$

$$H_{\dot{1}} = \partial/\partial\dot{x}, \quad H_{\dot{2}} = \partial/\partial\dot{y}.$$

Then the curvature R_{ρ} can be derived as

$$\begin{aligned}R_{\rho} =& [-T^{3}_{1}\mathrm{d}t\wedge\theta^{1} - T^{3}_{2}\mathrm{d}t\wedge\theta^{2} \\ &+ a(\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}(\dot{y}^2\theta^{1}\wedge\omega^{1} \\ &- \dot{x}\dot{y}\theta^{1}\wedge\omega^{2} - \dot{y}\dot{x}\theta^{2}\wedge\omega^{1} + \dot{x}^2\theta^{2}\wedge\omega^{2})]\otimes\partial/\partial z,\end{aligned}$$

where

$$T^{3}_{1} = a\boldsymbol{Z}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right),$$

$$T^{3}_{2} = a\boldsymbol{Z}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right).$$

The non-vanishing of the contraction of R_{ρ} indicates that the constraint is nonholonomic.

5. Concluding remarks

In this paper, we have utilized the connection theory on a fibre bundle to discuss the problem of integrability for nonlinear differential constraints. This method emphasizes the geometrical meaning of integrability conditions. Our construction is based on the fibrations of the configuration manifold M and the state constrained manifold N. The coordinate transformations respecting these fibrations can be visualized as a holonomy group of the Ehresmann connection, the algebra of which can be generated by the curvature form. It should be pointed out that the complete integrability mentioned above is in the Frobenius sense, unconcerned with the dynamical properties of the system. Otherwise, the Suslov's variation may accord with the commutativity of d and δ in some special cases. For example, the condition $T^{\varepsilon+\beta}_{\sigma} = 0$, which does not lead to the integrability of the constraints,

can be satisfied along the orbit of motion of the system in Appell-Hamel's example because the action is stationary along the orbit.

Acknowledgment

We are grateful to Professor Mei Feng-Xiang for his help and many valuable discussions.

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