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Generalized energy integral and reduction of rotational relativistic Birkhoffian system*

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We study the order reduction method of the rotational relativistic Birkhoffian equations. For a rotational relativistic autonomous Birkhoffian system, if the conservative law of the Birkhoffian holds, the conservative quantity can be called the generalized energy integral. Through the generalized energy integral, the order of the system can be reduced. If the rotational relativistic Birkhoffian system has a generalized energy integral, then the Birkhoffian equations can be reduced by at least two degrees and the Birkhoffian form can be kept. An example is given to illustrate the application of the result.

Keywords: rotational relativity, Birkhoffian equation, order reduction method, generalized energy integral

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1. Introduction

The integral method of dynamical equations of a complicated system is an important aspect of modern development in the field of mathematical mechanics and mathematical physics. In 1904, Whittaker initiated the order reduction method for a holonomic conservative system by using the energy integral.[1] After that, this method has always received attention from mathematicians, mechanicians and physicists. Since 1984, the Whittaker order reduction method has been generalized to the nonholonomic system, the generalized classical mechanics system, Vacco dynamical systems, the variable mass system, the non-inertial reference frame and the Birkhoffian system in Refs.[2]–[9]. But these studies have been confined to the classical mechanics systems.

The Birkhoffian system is more general than the Hamiltonian system.[10,11] The Hamiltonian dynamics has been extensively applied to the field of modern physics, so the Birkhoffian dynamics should play an important role in the field of modern physics. Recently, in Refs.[12]–[18] the theory of dynamics of the rotational relativistic Birkhoffian system has been constructed.

In this paper, we try to apply and extend the Whittaker order reduction method of the classical mechanics system to the rotational relativistic Birkhoffian system. Also, an example is given to illustrate the application of the result.

2. The generalized energy integral of the rotational relativistic Birkhoffian system

Here we study the fixed axis rotation of a mechanical system of N particles. At time t, the ith particle angular velocity is $\dot{\theta}_i$, the limiting angular velocity is Γ_i, the classical rotational moment of inertia is I_{oi}, and its relativistic rotational moment of inertia is[19,20]

$$I_i = \frac{I_{oi}}{\sqrt{1-\dot{\theta}_i^2/\Gamma_i^2}}, \quad (i=1,\cdots,N). \tag{1}$$

We can construct the Birkhoffian B^* and the Birkhoff functions $R_\nu^*(\nu=1,\cdots,2n)$ of a rotational relativistic system as

$$\begin{aligned} B^* &= B^*(I_i(t,a^\mu),t,a^\mu), \\ R_\nu^* &= R_\nu^*(I_i(t,a^\mu),t,a^\mu) \\ &(\nu,\mu=1,\cdots,2n), \end{aligned} \tag{2}$$

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and let

$$\tilde{B}^* = \tilde{B}^*(t, a^\mu) = B^*(I_i(t, a^\mu), t, a^\mu),$$

$$\tilde{R}^*_\nu = \tilde{R}^*_\nu(t, a^\mu) = R^*_\nu(I_i(t, a^\mu), t, a^\mu). \tag{3}$$

Then the Birkhoffian equations of the rotational relativistic system can be written in the form[12]

$$\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial t} = 0$$

$$(\nu, \mu = 1, \cdots, 2n), \tag{4}$$

where the repeated subscripts represent the summation. In general, it is supposed that the system (4) is non-singular, i.e.

$$\det(\omega^*_{\mu\nu}) \neq 0, \quad \omega^*_{\mu\nu} = \left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right). \tag{5}$$

If $\tilde{R}^*_\mu$ and$\tilde{B}^*$ do not include time t explicitly, the rotational relativistic Birkhoffian system (4) is autonomous. If $\tilde{R}^*_\mu$ do not include time t explicitly, the rotational relativistic Birkhoffian system (4) is semi-autonomous. For the autonomous and semi-autonomous systems, Eq.(4) has the simple form as follows

$$\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu} = 0$$

$$(\nu, \mu = 1, \cdots, 2n). \tag{6}$$

Multiplying the kth equation by $\dot{a}^k (k = 1, \cdots, 2n)$, and adding all these equations up, we can obtain

$$-\left(\frac{\partial \tilde{B}^*}{\partial a^1}\dot{a}^1 + \frac{\partial \tilde{B}^*}{\partial a^2}\dot{a}^2 + \cdots + \frac{\partial \tilde{B}^*}{\partial a^{2n}}\dot{a}^{2n}\right) = 0, \tag{7}$$

that is, $-\mathrm{d}\tilde{B}^*/\mathrm{d}t = 0$, or $\tilde{B}^*(a^1, a^2, \cdots, a^{2n}) = h$. Thus, we have:

Corollary 1 For a rotational relativistic autonomous Birkhoffian system, the conservative law of the Birkhoffian holds, i.e.

$$\tilde{B}^*(a^1, a^2, \cdots, a^{2n}) = h. \tag{8}$$

We call Eq.(8) the generalized energy integral of the system.

3. The order reduction method for the rotational relativistic Birkhoffian system

Using the generalized energy integral, we now reduce the order of Eq.(6).

Generally, we assume

$$\frac{\partial \tilde{B}^*}{\partial a^\mu} \neq 0. \tag{9}$$

From Eq.(8), we have

$$a^\mu = K(a^\nu, h) \ \ (\nu = 1, \cdots 2n; \nu \neq \mu), \tag{10}$$

$$\frac{\partial K}{\partial a^\nu} = -\frac{\partial \tilde{B}^*}{\partial a^\nu}\left(\frac{\partial \tilde{B}^*}{\partial a^\mu}\right)^{-1}. \tag{11}$$

Letting

$$a'^2 = \frac{\mathrm{d}a^2}{\mathrm{d}a^1} = \frac{\dot{a}^2}{\dot{a}^1}, \cdots,$$

$$a'^\nu = \frac{\mathrm{d}a^\nu}{\mathrm{d}a^1} = \frac{\dot{a}^\nu}{\dot{a}^1}, \cdots, \tag{12}$$

$$a'^{2n} = \frac{\mathrm{d}a^{2n}}{\mathrm{d}a^1} = \frac{\dot{a}^{2n}}{\dot{a}^1},$$

from the μth equation of the rotational relativistic Birkhoffian system (6), we obtain

$$\frac{1}{\dot{a}^1} = \left[\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\dot{a}'^\nu + \left(\frac{\partial \tilde{R}^*_1}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^1}\right)\right]\left(\frac{\partial \tilde{B}^*}{\partial a^\mu}\right)^{-1}$$

$$(\nu = 2, \cdots, 2n). \tag{13}$$

Substituting Eq.(13) into Eqs.(6) (except the first and μth equations), we obtain

$$\left[\frac{\partial \tilde{R}^*_\nu}{\partial a^\rho} - \frac{\partial \tilde{R}^*_\rho}{\partial a^\nu} + \frac{\partial K}{\partial a^\rho}\left(\frac{\partial \tilde{R}^*_\nu}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^\nu}\right)\right]a'^\nu$$
$$+ \left(\frac{\partial \tilde{R}^*_\mu}{\partial a^\rho} - \frac{\partial \tilde{R}^*_\rho}{\partial a^\nu}\right)a'^\mu + \frac{\partial K}{\partial a^\rho}\left(\frac{\partial \tilde{R}^*_1}{\partial a^\mu} - \frac{\partial \tilde{R}^*_\mu}{\partial a^1}\right)$$
$$+ \left(\frac{\partial \tilde{R}^*_1}{\partial a^\rho} - \frac{\partial \tilde{R}^*_\rho}{\partial a^1}\right) = 0$$

$$(\nu = 2, \cdots, 2n; \nu \neq \mu; \rho \neq 1, \mu). \tag{14}$$

Considering that

$$a'^\mu = \frac{\partial K}{\partial a^\nu}a'^\nu + \frac{\partial K}{\partial a^1} \ \ (\nu = 2, \cdots, 2n; \nu \neq \mu), \tag{15}$$

and making

$$\tilde{R}^{*'}_\nu(a^1, \cdots, a^{\mu-1}, a^{\mu+1}, \cdots, a^{2n})$$
$$= \tilde{R}^*_\nu(a^1, \cdots, a^{\mu-1}, K, a^{\mu+1}, \cdots, a^{2n}) \tag{16}$$

$$(\nu = 1, \cdots, 2n),$$

we have

$$\left[\left(\frac{\partial \tilde{R}^{*'}_\nu}{\partial a^\rho} + \frac{\partial \tilde{R}^{*'}_\mu}{\partial a^\rho}\frac{\partial K}{\partial a^\nu}\right) - \left(\frac{\partial \tilde{R}^{*'}_\rho}{\partial a^\nu} + \frac{\partial \tilde{R}^{*'}_\mu}{\partial a^\nu}\frac{\partial K}{\partial a^\rho}\right)\right]a'^\nu$$
$$+ \left(\frac{\partial \tilde{R}^{*'}_1}{\partial a^\rho} + \frac{\partial \tilde{R}^{*'}_\mu}{\partial a^\rho}\frac{\partial K}{\partial a^1}\right) - \left(\frac{\partial \tilde{R}^{*'}_\rho}{\partial a^1} + \frac{\partial \tilde{R}^{*'}_\mu}{\partial a^1}\frac{\partial K}{\partial a^\rho}\right) = 0. \tag{17}$$

Letting

$$\frac{\partial \tilde{\tilde{R}}_{\nu}^{*}}{\partial a^{\rho}} = \frac{\partial \tilde{R}_{\nu}^{*'}}{\partial a^{\rho}} + \frac{\partial \tilde{R}_{\mu}^{*'}}{\partial a^{\rho}} \frac{\partial K}{\partial a^{\nu}} \tag{18}$$

$$(\nu = 2, \cdots, 2n; \rho = 1, \cdots, 2n; \nu \neq \mu),$$

$$\frac{\partial \tilde{\tilde{R}}_{1}^{*}}{\partial a^{\rho}} = -\left(\frac{\partial \tilde{R}_{1}^{*'}}{\partial a^{\rho}} + \frac{\partial \tilde{R}_{\mu}^{*'}}{\partial a^{\rho}} \frac{\partial K}{\partial a^{1}} \right) \tag{19}$$

$$(\rho = 2, \cdots, 2n; \rho \neq \mu),$$

finally, we obtain

$$\left(\frac{\partial \tilde{\tilde{R}}_{\nu}^{*}}{\partial a^{\rho}} - \frac{\partial \tilde{\tilde{R}}_{\rho}^{*}}{\partial a^{\nu}} \right) a'^{\nu} - \left(\frac{\partial \tilde{\tilde{R}}_{1}^{*}}{\partial a^{\rho}} + \frac{\partial \tilde{\tilde{R}}_{\rho}^{*}}{\partial a^{1}} \right) = 0 \tag{20}$$

$$(\nu, \rho = 2, \cdots, 2n; \nu, \rho \neq \mu).$$

Through the above procedure, Eqs.(6) are reduced by two orders while the Birkhoffian form is still kept. After solving Eqs.(20), if Eqs.(8) and (13) are supplanted, we can obtain the solution of the original rotational relativistic Birkhoffian system (6). So we have:

Corollary 2 If a rotational relativistic Birkhoffian system has a generalized energy integral, then the Birkhoffian equations can be reduced by at least two orders and still keep Birkhoffian form.

4. Discussion

For the rotational relativistic autonomous Birkhoffian system (6), we let

$$a^{\mu} = \begin{cases} q^{\mu} & (\mu = 1, \cdots, n), \\ p_{\mu - n} & (\mu = n+1, \cdots, 2n), \end{cases}$$

$$\tilde{R}_{\mu}^{*} = \begin{cases} p_{\mu} & (\mu = 1, \cdots, n), \\ 0 & (\mu = n+1, \cdots, 2n), \end{cases} \tag{21}$$

$$\tilde{B}^{*} = H_r,$$

then the Birkhoffian tensors of the rotational relativistic system are

$$(\omega_{\mu\nu}^{*}) = \begin{bmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{bmatrix}. \tag{22}$$

We can obtain immediately the canonical equations of the rotational relativistic Hamiltonian system[12]

$$\dot{q}^{s} = \frac{\partial H_r}{\partial p_s}, \dot{p}_s = -\frac{\partial H_r}{\partial q^{s}} \quad (s = 1, \cdots, n) \tag{23}$$

and the rotational relativistic Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_r}{\partial \dot{q}^{s}} - \frac{\partial L_r}{\partial q^{s}} = 0 \quad (s = 1, \cdots, n), \tag{24}$$

where

$$H_r(t, q^{s}, p_s) = p_s \dot{q}^{s} - L_r, p_s = \frac{\partial L_r}{\partial \dot{q}^{s}},$$

$$L_r(t, q^{s}, \dot{q}^{s}) = T_r^{*} - V,$$

$$T_r^{*} = I_{oi} \Gamma_i^2 (1 - \sqrt{1 - \dot{\theta}_i^2 / \Gamma_i^2}) \tag{25}$$

$$(s = 1, \cdots, n; i = 1, \cdots, N).$$

Therefore, in this paper we give the generalized energy integral and reduction of the rotational relativistic Lagrangian system.

Under the classical approximation $\dot{\theta}_i \ll \Gamma_i$,

$$I_i = \frac{I_{oi}}{\sqrt{1 - \dot{\theta}_i^2 / \Gamma_i^2}} \approx I_{oi} \quad (i = 1, \cdots, N), \tag{26}$$

$$\tilde{B}^{*} = B^{*}(I_i(t, a^{\mu}), t, a^{\mu}) \approx B(t, a^{\mu}),$$

$$\tilde{R}_{\nu}^{*} = R_{\nu}^{*}(I_i(t, a^{\mu}), t, a^{\mu}) \approx R_{\nu}(t, a^{\mu}), \tag{27}$$

in this paper we give the generalized energy integral and reduction of the classical Birkhoffian system.

5. Example

As an example, we study a rotational relativistic Birkhoffian system, in which the Birkhoffian is

$$\begin{aligned} \tilde{B}^{*} =& I_{o1} \Gamma_1^2 (a^1)^2 + I_{o2} \Gamma_2^2 (a^2)^2 + I_{o3} \Gamma_3^2 (a^3)^2 \\ & - I_{o1} \Gamma_1^2 - I_{o2} \Gamma_2^2 - I_{o3} \Gamma_3^2, \end{aligned} \tag{28}$$

and the Birkhoff functions are

$$\tilde{R}_1^{*} = \tilde{R}_2^{*} = 0,$$

$$\tilde{R}_3^{*} = 2 I_{o1} \Gamma_1^2 a^1,$$

$$\tilde{R}_4^{*} = 2 I_{o2} \Gamma_2^2 a^2, \tag{29}$$

where

$$a^1 = (1 - \dot{q}_1^2 / \Gamma_1^2)^{-1/4},$$

$$a^2 = (1 - \dot{q}_2^2 / \Gamma_2^2)^{-1/4},$$

$$a^3 = (1 - \dot{q}_3^2 / \Gamma_3^2)^{-1/4}. \tag{30}$$

The Birkhoffian equations are

$$\dot{a}^3 - a^1 = 0, \dot{a}^4 - a^2 = 0,$$

$$I_{o1} \Gamma_1^2 \dot{a}^1 + I_{o3} \Gamma_3^2 a^3 = 0,$$

$$\dot{a}^2 = 0. \tag{31}$$

From corollary 1, we know that this Birkhoffian system has a generalized energy integral

$$\begin{aligned} & I_{o1} \Gamma_1^2 (a^1)^2 + I_{o2} \Gamma_2^2 (a^2)^2 + I_{o3} \Gamma_3^2 (a^3)^2 \\ & - I_{o1} \Gamma_1^2 - I_{o2} \Gamma_2^2 - I_{o3} \Gamma_3^2 = h = \text{const}. \end{aligned} \tag{32}$$

So according to corollary 2, the rotational relativistic Birkhoffian system can be reduced by two orders. From the reduced equations

$$\left(\frac{\partial \tilde{\tilde{R}}_{\nu}^{*}}{\partial a^{\rho}}-\frac{\partial \tilde{\tilde{R}}_{\rho}^{*}}{\partial a^{\nu}}\right)a^{'\nu}-\left(\frac{\partial \tilde{\tilde{R}}_{1}^{*}}{\partial a^{\rho}}+\frac{\partial \tilde{\tilde{R}}_{\rho}^{*}}{\partial a^{1}}\right)=0 \qquad (33)$$

$$(\nu,\rho=2,4),$$

we obtain

$$a'^{2}=0,$$

$$a'^{4}=I_{o1}\Gamma_{1}^{2}a^{2}[(h-I_{o1}\Gamma_{1}^{2}(a^{1})^{2}-I_{o2}\Gamma_{2}^{2}(a^{2})^{2}+I_{o1}\Gamma_{1}^{2}+I_{o2}\Gamma_{2}^{2}+I_{o3}\Gamma_{3}^{2})I_{o3}\Gamma_{3}^{2}]^{-1/2}. \qquad (34)$$

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