Vol 11 No 10, October 2002
1009-1963/2002/11(10)/1028-05

Chinese Physics

© 2002 Chin. Phys. Soc.
and IOP Publishing Ltd

Position-dependent dynamics of a trapped ion in a standing wave laser*

Fang Mao-Fa(方卯发)

Department of Physics, Hunan Normal University, Changsha 410081, *China*

(Received 28 January 2002; revised manuscript received 11 April 2002)

We have investigated the position-dependent dynamics of a trapped ion in a standing wave laser by transforming it to the Jaynes–Cummings-type system under the Lamb–Dicke limit. A variety of novel phenomena are exhibited, e.g. periodic collapse and revival features and long-time scaled revivals of the ionic inversion, depending on its position in the standing wave. Our result provides a way of producing a system equivalent to the two-photon Jaynes–Cummings model in the trapped ion system, with its exact periodicities.

Keywords: position-dependent dynamics, trapped ion, collapse and revival
PACC: 4250V, 3290, 0365

Trapped ions interacting with laser beams[1,2] have become an extremely interesting system for the investigation of fundamental physics, for instance, the generation of nonclassical states of the vibration motion of the ion[3−7] as well as for potential applications such as precision spectroscopy,[6] quantum computation,[8] and quantum state reconstruction.[9−10] Recently, Sϕrensen and Mϕlmer[11] have proposed a novel scheme for realizing quantum computation in ion traps via virtual vibrational excitations. This scheme does not use the motional mode as the data bus, and it is insensitive to the vibrational states. Sϕrensen and Mϕlmer[12] have also proposed a scheme for the generation of multiparticle entanglement in ion traps also without the requirement of the full control of the ion motion, which has been accomplished in experiment.[13] These works show that the trapped ion system is good for quantum information processing.[14] Although the theoretical treatment of the interaction of a trapped ion with one or several laser beams constitutes a complicated problem, being the full Hamiltonian and highly nonlinear, it can be greatly simplified under certain limiting conditions, such as the Lamb–Dicke limit and the Strong confinement limit. For example, Cirac *et al*[3] have shown that the dynamics of a trapped two-level ion, at the node of a standing wave, is described by the Jaynes–Cummings model (JCM); later Wu and Yang,[15] and Alam and Bentley[16] demonstrated that the dynamics of a trapped ion in any position of a standing wave can also be described by the JCM on the ionic dressed basis. In this paper, we introduce a unitary transformation which transforms the Hamiltonian of the trapped ion to that of the normal JCM on the bare basis of the trapped ion. By using this transformation, we may obtain the general evolution of the state vector of the trapped ion system.

Quantum collapses and revivals are directly related to the discreteness of the vibrational states and provide a novel means of measuring the statistics of the quantum motion of the trapped ion. The phenomenon was theoretically predicted by Blockley *et al*[17] in a trapped ion interacting with a running wave, and experimentally observed by Meekhof *et al*[18] in NIST. For a single trapped ion interacting with a standing wave field, Cirac *et al*[19] and Vogel and de Matos Filho[20] have also reported quantum collapse and revival features. However, these results are obtained only under the condition that the trapped ion is located close to the node of a standing-wave.[19,20] In this paper, we investigate quantum collapses and revivals of a trapped ion in any position of a standing wave.

We consider a single two-level ion trapped in a

*Project supported by the National Natural Science Foundation of China (Grant No 19874020) and by the Natural Science Foundation of Hunan Province, China (Grant No 01JJy3030).

harmonic potential of frequency ν and interacting with a standing wave. In the situation where the trap frequency ν is much greater than the ionic decay rate, the Hamiltonian is[1]

$$H = \nu a^+ a + \frac{\Delta}{2}\sigma_z + \frac{\Omega_0}{2}\sigma_x \cos[\eta(a + a^+) + \chi]. \quad (1)$$

Here, a and a^+ are the annihilation and creation operators for the phonons or the vibrational states of the trap, respectively; $\Delta = \omega_0 - \omega_l$ is the detuning between the transition and laser frequencies; $\sigma_x = \sigma_{21} + \sigma_{12}$ and $\sigma_z = \sigma_{22} - \sigma_{11}$ are the two-level polarization and inversion operators; Ω_0 is the Rabi frequency proportional to the amplitude of the standing wave; η is the Lamb–Dicke parameter; and χ accounts for the relative position of the ion in the standing wave. In particular, $\chi = \pi/2$ corresponds to an ion centred at a node of the standing wave. We expand Eq.(1) in powers of the Lamb–Dicke parameter η to obtain

$$H = \nu a^+ a + \frac{\Delta}{2}\sigma_z + \frac{\Omega_0}{2}\cos(\chi)\sigma_x - \eta\frac{\Omega_0}{2}\sin(\chi)\sigma_x(a + a^+) + O(\eta^2), \quad (2)$$

where higher-order terms of η are negligible in the Lamb–Dicke limit. We introduce a unitary transform operator

$$T = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}, \quad (3)$$

with

$$x = \sqrt{\frac{\Delta + \Omega}{2\Omega}}, \qquad y = \frac{\Omega_0 \cos\chi}{\sqrt{2\Omega(\Delta + \Omega)}},$$
$$\Omega = \sqrt{\Delta^2 + \Omega_0^2 \cos^2\chi}. \quad (4)$$

The Hamiltonian (2) is transformed by T to

$$H' = THT^+ = \nu a^+ a + \frac{\Omega}{2}\sigma_z\left[1 - \frac{\eta\Omega_0^2 \sin 2\chi}{2\Omega^2}(a + a^+)\right] + \frac{\eta\Omega_0\Delta\sin\chi}{2\Omega}\sigma_x(a + a^+). \quad (5)$$

By carrying out a rotating-wave approximation on Eq.(5), we then find

$$H' = H_{\rm JCM} = \nu a^+ a + \frac{\Omega}{2}\sigma_z + g(\sigma_{21}a + \sigma_{12}a^+), \quad (6)$$

where $g = \eta\dfrac{\Omega_0\Delta}{2\Omega}\sin\chi$. Therefore, we can directly use the results of the normal JCM in studying the dynamics of the trapped ion in any position of the standing wave. For example, the state vector $|\Psi(t)\rangle$ of the trapped ion system, governed by Hamiltonian (1), can be obtained by transforming the state vector $|\Psi_{\rm JCM}(t)\rangle$ of the well-studied JCM[21] through the unitary transform operator T: $|\Psi(t)\rangle = T^+|\Psi_{\rm JCM}(t)\rangle$.

For the ion initially in the ground state $|1\rangle$ and the phonons in a coherent state

$$|\Psi_{\rm p}(0)\rangle = \sum_{n=0}^{\infty} P_n|n\rangle,$$

with

$$P_n = \exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^n}{\sqrt{n}}, \quad (7)$$

where $\alpha = \sqrt{\bar{n}}\exp(\mathrm{i}\beta)$, $\bar{n}$ is the initial average phonon number, and β is the phase of the phonon coherent state. The time-dependent state vector $|\Psi(t)\rangle$, after a straightforward calculation, can be expressed in the form

$$|\Psi(t)\rangle = |C(t)\rangle|2\rangle + |S(t)\rangle|1\rangle, \quad (8)$$

where

$$|C(t)\rangle = \sum_{n=0}^{\infty} P_n \exp(-\mathrm{i}n\nu t)\Big\{\exp\left(-\mathrm{i}\frac{\nu t}{2}\right) \times [xy\zeta_n|n\rangle - \mathrm{i}y^2\eta_n|n+1\rangle] - \exp\left(\mathrm{i}\frac{\nu t}{2}\right)[xy\zeta_{n-1}|n\rangle - \mathrm{i}x^2\eta_{n-1}|n-1\rangle]\Big\}, \quad (9)$$

$$|S(t)\rangle = \sum_{n=0}^{\infty} P_n \exp(-\mathrm{i}n\nu t)\Big\{\exp\left(-\mathrm{i}\frac{\nu t}{2}\right) \times [y^2\zeta_n|n\rangle + \mathrm{i}xy\eta_n|n+1\rangle] + \exp\left(\mathrm{i}\frac{\nu t}{2}\right)[x^2\zeta_{n-1}|n\rangle + \mathrm{i}xy\eta_{n-1}|n-1\rangle]\Big\}, \quad (10)$$

with

$$\zeta_n = \cos A_n t - \mathrm{i}\frac{\delta}{2}\frac{\sin A_n t}{A_n}, \qquad \eta_n = \frac{g\sqrt{n+1}\sin A_n t}{A_n},$$
$$A_n = \sqrt{\left(\frac{\delta}{2}\right)^2 + g^2(n+1)}, \qquad \delta = \Omega - \nu. \quad (11)$$

In what follows, we explore the position-dependent dynamics of the trapped ion in terms of the quantum collapses and revivals of the ionic inver-

sion, which take the form

$$\begin{aligned} W(t) =& \langle \Psi(t)|S_z|\,\Psi(t)\rangle \\ =& \frac{1}{2}\sum_{n=0}^{\infty}\Big\{(x^2-y^2)\big[y^2(|\zeta_n|^2-|\eta_n|^2) \\ & - x^2(|\zeta_{n-1}|^2-|\eta_{n-1}|^2)\big]|P_n|^2 \\ & - 4xy\mathrm{Re}\{\exp(\mathrm{i}\nu t) \\ & \times [\mathrm{i}(y^2\zeta_n^*\eta_{n-1}P_{n-1}P_n^* - x^2\zeta_{n-1}^*\eta_n P_n P_{n+1}^*) \\ & + xy(\zeta_n^*\zeta_{n-1}^*|P_n|^2 + \eta_n\eta_{n-1}P_{n-1}P_{n+1}^*)]\}\Big\}, \end{aligned} \tag{12}$$

where $S_z = \frac{1}{2}(|2\rangle\langle 2| - |1\rangle\langle 1|)$ is the inversion operator of the trapped ion. The expectation value for S_z reflects the occupation difference between the ground state $|1\rangle$ and the excited state $|2\rangle$. Figure 1 presents numerical results for the ionic inversion $W(t)$ for $\Omega_0 = 6\pi\times 10^5\mathrm{s}^{-1}$,[19] $\Delta = \Omega_0/20, \eta = 0.05$, $\bar{n} = 25$ and $\delta = -\Omega$. In Fig.1(a), we have assumed that the trapped ion is located at the node of the standing wave ($\chi = \pi/2$). Obviously, the time evolution of the ionic inversion $W(t)$ exhibits the collapse–revival profile, which is similar to that in the normal JCM.[21] If we move the trapped ion away from the node, say $\chi = \pi/4$ (in Fig.1(b)) and $\chi = \pi/10$ (Fig.1.(c)) then the quantum collapses and revivals of the ionic inversion can well demonstrate a periodicity as time passes. Besides, this also shows long timescale revivals. It is evident from this graph that the quantum dynamics of the trapped ion is highly dependent on its position in the standing wave.

We should point out that the periodic collapse and revival feature of the trapped ion occurs only when $\delta = -\Omega$. Otherwise, no periodic profile shows up in the time evolution of the ionic inversion, although it still exhibits quantum collapses and revivals. See, for instance, in Fig.2, where we set $\delta = \Omega - \nu = 0$, and the other parameters are same as in Fig.1.

The phenomenon of collapse and revival has the same origin as that exhibited in the standard JCM.[21] From Eq.(12), one can see that the inversion consists of summations over the number n of phonons. Each term in the summations represents Rabi oscillations for a definite value of n. The phonon distribution function $P_{nn}(0) = |P_n|^2$ determines the relative weight for each value of n. If the atom is prepared in a defined state at the initial time ($t = 0$), all the terms in the summations are correlated. As time passes, the Rabi oscillations associated with different excitations have different frequencies and are therefore uncorrelated. This leads to a collapse of the inversion. As the time further increases, the correlation is restored and revival occurs. The behaviour continues and an infinite sequence of revivals may be shown.

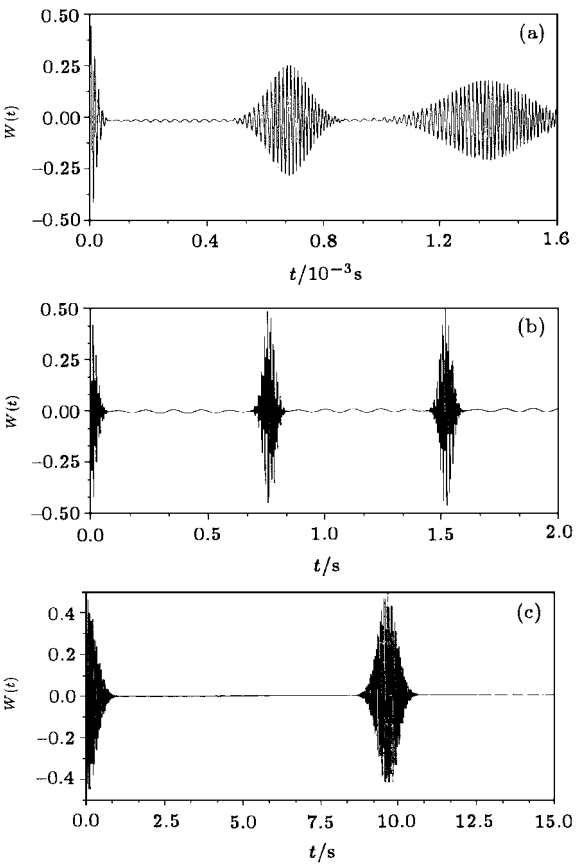


Fig.1. The time evolution of the ionic inversion $W(t)$ of a trapped ion in a standing wave under the off-resonance condition. The ion is initially in the ground state $|1\rangle$ and phonons are in the coherent state with average phonon number $\bar{n} = 25$, $\Omega_0 = 6\pi\times 10^5\mathrm{s}^{-1}$, $\Delta = \Omega_0/20$, and $\eta = 0.05, \delta = -\Omega$. (a) The trapped ion is located at the node of the standing wave, $\chi = \pi/2, \beta = 0$. (b) The trapped ion is placed from away from the node of the standing wave, $\chi = \pi/4, \beta = 0$. (c) The trapped ion deviates from the node of the standing wave, $\chi = \pi/10, \beta = 0$.

To further explore the dynamical behaviour of the ionic inversion, we here present an analytical investigation of Eq.(12). For the case of the trapped ion locating at the node of the standing wave ($\chi = \pi/2$), $x = 1$ and $y = 0$, Eq.(12) reduces to the form

$$\begin{aligned} W(t) =& -\frac{1}{2}\sum_{n=0}^{\infty}(|\zeta_{n-1}|^2 - \eta_{n-1}^2)|P_n|^2 \\ \approx& \begin{cases} -\frac{1}{2}\sum_{n=0}^{\infty}\cos[2g\sqrt{n}(1+\delta^2/(4g^2n))t]|P_n|^2 \\ \text{for } \delta = -\Omega; \\ -\frac{1}{2}\sum_{n=0}^{\infty}\cos(2g\sqrt{n}t)|P_n|^2 \\ \text{for } \delta = 0. \end{cases} \end{aligned} \tag{13}$$

This formalism is the same as the expression for the inversion in the normal JCM and therefore it exhibits the well-known quantum collapses and revivals for suitable choices of P_n. The revival time $T_{\rm RJ}$ can be obtained from Eq.(13)

$$T_{\rm RJ} \approx \begin{cases} \dfrac{2\pi\bar{n}^{1/2}}{g} \Big/ \left(1+\dfrac{\delta^2}{4g^2\bar{n}}\right) & \text{for } \delta=-\Omega; \\ \dfrac{2\pi\bar{n}^{1/2}}{g} & \text{for } \delta=0. \end{cases} \tag{14}$$

As a result, the revival time $T_{\rm RJ} \approx 6.41\times10^{-4}$s in the case of $\delta=-\Omega$, while $T_{\rm RJ}=6.67\times10^{-4}$s for $\delta=0$. Therefore, the time evolution of the ionic inversion $W(t)$ for both cases is almost identical (see Figs.1(a) and 2(a), for example).

When the trapped ion deviates from the node of the standing wave, for instance, $\chi=\pi/4$ and $\Delta \ll \Omega$, then $x\approx y$ and $W(t)$ has the approximate form

$$\begin{aligned} W(t) \approx & -2x^2y^2\sum_{n=0}^{\infty}|P_n|^2\Big\{\mathrm{Re}\Big\{\exp(\mathrm{i}\nu t) \\ & \times\Big[-\mathrm{i}\frac{g\sqrt{n}}{A_{n-1}}\sin(A_n-A_{n-1})t \\ & +\cos A_n t\cos A_{n-1}t \\ & -\frac{\delta^2}{4A_nA_{n-1}}\sin A_nt\sin A_{n-1}t \\ & +\mathrm{i}\Big(\frac{\delta\sin A_nt\cos A_{n-1}t}{2A_n} \\ & +\frac{\delta\cos A_nt\sin A_{n-1}t}{2A_{n-1}}\Big) \\ & +\frac{g^2n\sin A_nt\sin A_{n-1}t}{A_nA_{n-1}}\Big]\Big\}\Big\}. \end{aligned} \tag{15}$$

Here we have assumed that $x\approx y$ and $n\approx\bar{n}\gg1$.

For the parameters given in Fig.1, we may make further approximations: $A_n\approx A_{n-1}$, $\delta/2A_n\approx-1$, $\delta^2/4A_nA_{n-1}\approx1$ and $g^2/4A_n^2\approx0$ for the case of $\delta=-\Omega$; and $g^2\bar{n}/A_nA_{n-1}\approx1$ for the case of $\delta=0$. Then Eq.(15) can be further reduced to

$$W(t)\approx\begin{cases} -2x^2y^2\displaystyle\sum_{n=0}^{\infty}\cos\big[3\Omega+2g^2(2n+1)/\Omega\big]t \\ \text{for } \delta=-\Omega; \\ -2x^2y^2\displaystyle\sum_{n=0}^{\infty}\cos\big[\Omega-g/2\sqrt{n}\big]t \\ \text{for } \delta=0. \end{cases} \tag{16}$$

The corresponding revival time $T_{\rm R}$ can be obtained as

$$T_{\rm R}\approx\begin{cases} \pi\Omega/g^2 & \text{for } \delta=-\Omega; \\ 8\pi\bar{n}^{3/2}/g & \text{for } \delta=0. \end{cases} \tag{17}$$

It is clear from Eq.(16) that, under the condition $\delta=-\Omega$, the ionic inversion $W(t)$ undergoes Rabi oscillations with a Rabi frequency proportional to n. This is similar to the case in the two-photon JCM and leads to the periodical time evolution of the ionic inversion, as in Figs.1(b) and 1(c). However, for the case of $\delta=0$, the Rabi oscillation frequency of ionic inversion $W(t)$ is proportional to $1/\sqrt{n}$. This is similar to the case in the one-photon JCM. Thus, the time behaviour of the ionic inversion $W(t)$ is not periodical in this case.

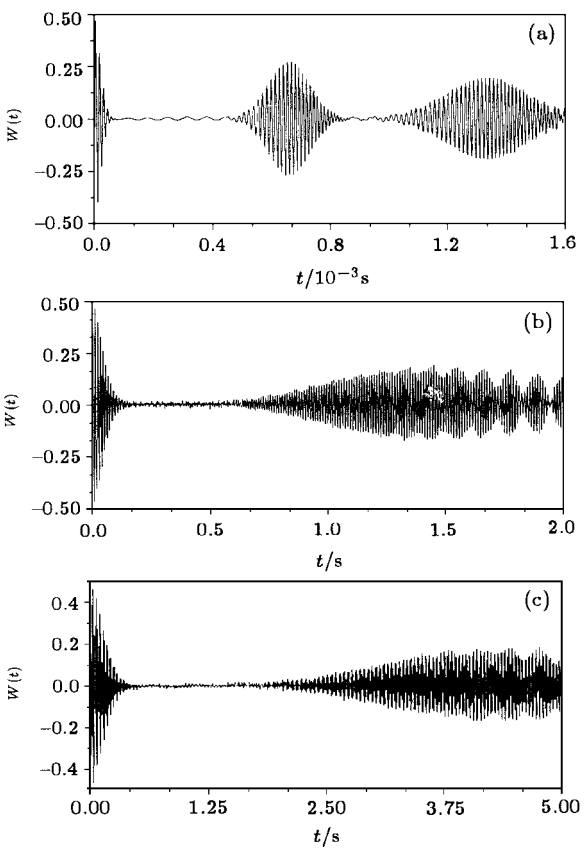


Fig.2. The time evolution of the ionic inversion $W(t)$ of a trapped ion in a standing wave under the resonance condition. The choices of parameters are the same as in Fig.1 but $\delta=0$. (a) The trapped ion is located at the node of the standing wave, $\chi=\pi/2$, $\beta=0$. (b) The trapped ion is placed away from the node of the standing wave, $\chi=\pi/4$, $\beta=0$. (c) The trapped ion deviates from the node of the standing wave, $\chi=\pi/10$, $\beta=0$.

The approximate analysis gives us quantitatively the timescales of the evolution of the ionic inversion $W(t)$ varying with ionic positions. For example, in the case of $\delta=-\Omega$, the revival time $T_{\rm RJ}=\dfrac{2\pi\bar{n}^{1/2}}{g}\Big/\left(1+\dfrac{\delta^2}{4g^2\bar{n}}\right)\approx6.41\times10^{-4}$s for $\chi=\pi/2$, and the revival time $T_{\rm R}=\pi\Omega/g^2\approx7.61\times10^{-1}$s for $\chi=\pi/4$. Whereas in the case of $\delta=0$, the revival time $T_{\rm RJ}=2\pi\bar{n}^{1/2}/g\approx6.67\times10^{-4}$s for $\chi=\pi/2$, and

the revival time $T_{\rm R} = 8\pi\bar{n}^{3/2}/g \approx 1.34$s for $\chi = \pi/4$. Therefore, when the trapped ion is not situated at the node of the standing wave, the timescales of the quantum collapses and revivals of the ionic inversion are much longer. These results are almost in agreement with the observed revival times in Figs.1 and 2.

We can qualitatively interpret the long timescale revivals of the ionic inversion when the trapped ion is placed away from the node of the standing wave in terms of the effective coupling constant g. It is well known that the revival time is proportional to g^{-1} in the standard JCM,[21] which is also qualitatively applicable to our trapped ion system. As is known, the effective coupling coefficient, $g = \eta\Omega_0\Delta\sin\chi/2\Omega$, is strongly position-dependent. When $\chi = \pi/2$, the coupling is maximal and thus the revival time is shortest. Otherwise, the ionic inversion will revive at a longer timescale. As can be seen, for example, in Figs.1(c) and 2(c), the revival time for $\chi = \pi/10$ is much longer than that for $\chi = \pi/2$ as in Figs.1(a) and 2(a). The features of the long timescale revivals of the ionic inversion are a unique manifestation of the position dependence of the ionic dynamics.

In conclusion, we have investigated the position-dependent dynamics of a trapped ion in a standing wave laser by transforming it to the Jaynes–Cummings-type system in the Lamb–Dicke limit. A variety of novel phenomena is exhibited, for example, periodic collapse and revival features and long timescale revivals of the ionic inversion, depending on its position in the standing wave. It is well known that there are exact periodical dynamics in the two-photon JCM in quantum optics. Therefore, our result provides a way of producing a system equivalent to the two-photon JCM in a trapped ion system, with its exact periodicities under the condition $\delta = -\Omega$. It is noted that the experimental realization of the two-photon JCM is somewhat difficult in the experiment of cavity quantum electrodynamics, due to the effects of the dissipative processes in the system with atomic spontaneous and cavity damping. While in the trapped ion system, the dissipative effects are essentially negligible as the coupling between the vibrational modes and the environment is extremely weak, and spontaneous emission may be avoided or 'controlled' by using electric–dipole forbidden transitions or Raman transitions between metastable ground states of the trapped ion. Hence, our result is important for the experimental realization of the the dynamics with periodic collapse and revival.

References

[1] Wineland D J C. Monroe, Itano W M, Leibfried D, King B E and Meekhof D M 1998 *J. Res. Natl. Inst. Stand.* **103** 259

[2] Wieman C E, Pritchard D E and Wineland D J 1999 *Rev. Mod. Phys.* **71** S253

[3] Heinzen D J and Wineland D J 1990 *Phys. Rev.* A **42** 2977
Cirac J I, Blatt R, Parins A S and Zoller P 1993 *Phys. Rev. Lett.* **70** 762

[4] Heinzen D J and Wineland D J 1990 *Phys. Rev.* A **42** 2977
De Matos Fiho R L and Vogel W 1996 *Phys. Rev. Lett.* **76** 608

[5] Gerry C C, Gou S C and Steinbach J 1997 *Phys. Rev.* A **55** 63
Gerry C C, Gou S C and Steinbach J 1997 *Phys. Rev.* A **55** 2478

[6] Wineland D J, Bollinger J J, Itano W M, Moore F L and Heinzen D J 1992 *Phys. Rev.* A **46** R6797

[7] Jiang Y R 2001 *Chin. Phys.* **10** 202
Fang M F and Liu X 2001 *Chin. Phys.* **10** 720
Feng M, Guo X L, Zhu X W and Lei Y 1998 *Acta Phys. Sin.* (Overseas Edition) **7** 488

[8] Cirac J I and Zoller P 1995 *Phys. Rev. Lett.* **74** 4091

[9] Wallentowitz S and Vogel W 1995 *Phys. Rev. Lett.* **75** 2932
Poyatos J F, Walser R, Cirac J I and Zoller P 1996 *Phys. Rev.* A **53** R1966
Helon D J and Miburn G J 1996 *Phys. Rev.* **54** R25

[10] Leibfried D, Meekhof D M, King B E, Monroe C, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **77** 4281

[11] Sørensen A and Mølmer K 1999 *Phys. Rev. Lett.* **82** 1971

[12] Sørensen A and Mølmer K 1999 *Phys. Rev. Lett.* **82** 1835

[13] Sachett C A 2000 *Nature* **404** 256

[14] Zheng S B and Guo G C 2000 *Phys. Rev. Lett.* **85** 2392
Guo G P, Li C F, Li J and Guo G C 2002 *Phys. Rev.* A **65** 032144; *Preprint* quant-ph/0105123

[15] Wu Y and Yang X X 1997 *Phys. Rev. Lett.* **78** 3086

[16] Alam S and Bentley C 1997 *Preprint* Physics/9707004

[17] Blockley C A, Walls D F and Risken H 1992 *Europhys. Lett.* **17** 509

[18] Meekhof D M, Monroe C, King B E, Itano W M and Wineland D J 1996 *Phys. Rev. Lett.* **76** 1796

[19] Cirac J I, Blatt R, Parkins R S and Zoller P 1994 *Phys. Rev.* A **49** 1202

[20] Vogel W and de Matos Filho R L 1995 *Phys. Rev.* A **52** 4214

[21] Scully M O and Zubairy M S 1997 *Quantum Optics* (Cambridge: Cambridge University Press) chap 6 and references therein