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Bifurcations of a parametrically excited oscillator with strong nonlinearity*

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A parametrically excited oscillator with strong nonlinearity, including van der Pol and Duffing types, is studied for static bifurcations. The applicable range of the modified Lindstedt–Poincaré method is extended to 1/2 subharmonic resonance systems. The bifurcation equation of a strongly nonlinear oscillator, which is transformed into a small parameter system, is determined by the multiple scales method. On the basis of the singularity theory, the transition set and the bifurcation diagram in various regions of the parameter plane are analysed.

Keywords: strongly nonlinear oscillator, parameter excitation, bifurcation
PACC: 0574, 0520D, 0365D, 1110L

1. Introduction

Classical perturbation methods, that include the Lindstedt–Poincaré (LP), Krylov–Bogoliubov–Mitropolski (KBM) and multiple scales methods, are widely used in the study of nonlinear phenomena, but are limited to weakly nonlinear systems. The study of bifurcation problems of the parametrically excited oscillator on the basis of determining the bifurcation equation has led to better results,[1−3] but these are also limited to weakly nonlinear systems. The modified Lindstedt–Poincaré (MLP) method for the analysis of strongly nonlinear oscillation governed by an equation of motion containing a parameter ε, which need not be small, was presented by Cheung *et al.*[4] In the present paper, we extend the applicable range of the MLP method to 1/2 subharmonic resonance systems.[5,6] A parametrically excited oscillator with strong nonlinearity, that includes van der Pol and Duffing types, is studied for static bifurcations. The bifurcation equation is obtained for the 1/2 subharmonic resonance case. On the basis of the singularity theory,[8−10] the bifurcation equation is reduced to Golubitsky–Schaeffer (GS) normal form. After unfolding the transition set, the bifurcation diagram in various regions of the parameter plane is analysed.

2. Parameter transformation

We consider the strongly nonlinear oscillator

$$\ddot{x} + x + \varepsilon[(\mu + \eta x^2)\dot{x} + \cos 2t \cdot x + \beta x^3] = 0, \quad (1)$$

where ε is not a small parameter. The parametric excitation is near 1/2 subharmonic resonance. A new parameter is defined

$$\alpha = \frac{\varepsilon\omega_1}{(2\omega_0)^2 + \varepsilon\omega_1} = \frac{\varepsilon\omega_1}{4 + \varepsilon\omega_1}. \quad (2)$$

Let

$$\tau = \omega t. \quad (3)$$

Substituting Eq.(3) into Eq.(1), we obtain

$$\omega^2 x'' + x + \varepsilon[(\mu + \eta x^2)\omega x' + x\cos\tau + \beta x^3] = 0, \quad (4)$$

where $x' = \dfrac{\mathrm{d}x}{\mathrm{d}\tau}$ and $x'' = \dfrac{\mathrm{d}^2x}{\mathrm{d}\tau^2}$. Solving Eq.(2) yields

$$\varepsilon = \frac{4\alpha}{\omega_1(1-\alpha)}. \quad (5)$$

The unknown ω is usually expanded in powers of α, that is

$$\begin{aligned}\omega^2 =& (2\omega_0)^2 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \\ =& (4 + \varepsilon\omega_1)\left[1 + \frac{1}{4 + \varepsilon\omega_1}(\varepsilon^2\omega_2 + \cdots)\right] \\ =& \frac{4}{1-\alpha}(1 + \delta_2\alpha^2 + \delta_3\alpha^3 + \cdots), \end{aligned} \quad (6)$$

then

$$\omega = 2\left[1 + \frac{1}{2}\alpha + \left(\frac{3}{8} + \frac{\delta_2}{2}\right)\alpha^2 + \cdots\right]. \quad (7)$$

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Expanding x into a power series with respect to α, we let

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \cdots \tag{8}$$

Substituting Eqs.(5)–(8) into Eq.(4) and equating the coefficients of like terms of α, we have

$$x_0'' + \frac{1}{4}x_0 = 0, \tag{9}$$

$$\begin{aligned} x_1'' + \frac{1}{4}x_1 =& \frac{1}{4}x_0 - \frac{2}{\omega_1}(\mu + \eta x_0^2)x_0' \\ & - \frac{1}{\omega_1}x_0\cos\tau - \frac{1}{\omega_1}\beta x_0^3, \\ & \cdots\cdots \end{aligned} \tag{10}$$

The solution of Eq.(9) with the initial condition $x_0(0) = a_0$ is

$$x_0 = a_0\cos\frac{\tau}{2} + b_0\sin\frac{\tau}{2}. \tag{11}$$

Substituting Eq.(11) into Eq.(10) and eliminating the secular terms, we obtain

$$\begin{aligned} & a_0\omega_1 - 4\mu b_0 - 2a_0 - 3\beta a_0(a_0^2 + b_0^2) \\ & - \eta b_0(a_0^2 + b_0^2) = 0, \\ & b_0\omega_1 + 4\mu a_0 + 2b_0 - 3\beta b_0(a_0^2 + b_0^2) \\ & + \eta a_0(a_0^2 + b_0^2) = 0. \end{aligned} \tag{12}$$

Solving Eq.(12), we obtain the following result

$$\omega_1 = 3\beta(a_0^2 + b_0^2) + \frac{2}{a_0^2 + b_0^2}(a_0^2 - b_0^2), \tag{13}$$

and b_0 can be determined as

$$4\mu(a_0^2 + b_0^2) + 4a_0b_0 + \eta(a_0^2 + b_0^2)^2 = 0. \tag{14}$$

Thus ω_1 is known and the new expansion parameter α is defined.

3. Bifurcation equation

According to the procedure in the above section, a strongly nonlinear system with a large parameter ε is transformed into a small system with respect to α. Equation (1) can be written as

$$\begin{aligned} & (1 + \delta_2\alpha^2 + \cdots)x'' + \frac{1-\alpha}{4}x + \frac{\alpha}{\omega_1} \\ & \times \left[(\mu + \eta x^2)2\left(1 + \frac{1}{2}\alpha + \cdots\right)x' \right. \\ & \left. + x\cos\tau + \beta x^3\right] = 0. \end{aligned} \tag{15}$$

In this section we study Eq.(15) by using the multiple method.[7] Expanding x in powers of α, we let

$$x = x_0(T_0, T_1) + ax_1(T_0, T_1) + \cdots, \tag{16}$$

where $T_0 = \tau, T_1 = \alpha\tau$. The differential operators are given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau} =& \frac{\partial}{\partial T_0} + \alpha\frac{\partial}{\partial T_1} + \cdots \\ =& D_0 + \alpha D_1 + \cdots, \\ \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} =& D_0^2 + 2\alpha D_0D_1 + \cdots \end{aligned} \tag{17}$$

Substituting Eqs.(16) and (17) into Eq.(15) and equating the coefficients of like terms of α, we can obtain the following set of linear differential equations

$$D_0^2x_0 + \frac{1}{4}x_0 = 0, \tag{18}$$

$$\begin{aligned} & D_0^2x_1 + \frac{1}{4}x_1 \\ =& -2D_0D_1x_0 + \frac{1}{4}x_0 - \frac{2}{\omega_1}(\mu + \eta x_0^2)D_0x_0 \\ & - \frac{1}{\omega_1}x_0\cos T_0 - \frac{1}{\omega_1}\beta x_0^3 \\ & \cdots\cdots. \end{aligned} \tag{19}$$

The general solution of Eq.(18) is

$$x_0 = A\mathrm{e}^{\mathrm{i}T_0/2} + \overline{A}\mathrm{e}^{-\mathrm{i}T_0/2}. \tag{20}$$

Substituting Eq.(20) into Eq.(19) and eliminating the secular terms yields

$$\begin{aligned} D_1A =& -\frac{1}{\omega_1}\mu A - \frac{1}{\omega_1}\eta A^2\overline{A} - \frac{1}{4}\mathrm{i}A \\ & + \frac{1}{2\omega_1}\mathrm{i}\overline{A} + \frac{3}{\omega_1}\mathrm{i}\beta A^2\overline{A}. \end{aligned} \tag{21}$$

We let $A = \frac{1}{2}a\mathrm{e}^{\mathrm{i}\varphi}$, where $a = a(T_1)$ and $\varphi = \varphi(T_1)$. Equation (21) transforms to

$$\begin{aligned} a' &= -\frac{1}{\omega_1}\mu a - \frac{1}{4\omega_1}\eta a^3 + \frac{1}{2\omega_1}a\sin 2\varphi, \\ a\varphi' &= -\frac{1}{4}a + \frac{3}{4\omega_1}\beta a^3 + \frac{1}{2\omega_1}a\cos 2\varphi. \end{aligned} \tag{22}$$

At the singular point $a' = \varphi' = 0$, eliminating φ from Eq.(22) we have the bifurcation equation

$$\begin{aligned} & (9\beta^2 + \eta^2)a^4 + 2(4\mu\eta - 3\beta\omega_1)a^2 \\ & - (4 - 16\mu^2 - \omega_1^2) = 0. \end{aligned} \tag{23}$$

Equation (23) is a set of parameter values at which system (1) may be structurally unstable. The method is suitable for other problems.[11–15].

4. Transition set and bifurcation diagram

We now present the analysis of static bifurcation in terms of the singularity theory. Summarizing the possible results, we obtain the following.

(i) When $4\mu\eta - 3\beta\omega_1 = 0$ and $9\beta^2 + \eta^2 \neq 0$, the bifurcation Eq.(23) can be written as

$$a^4 - \frac{4 - 16\mu^2 - \omega_1^2}{9\beta^2 + \eta^2} = 0 \tag{24}$$

or

$$a^4 - \lambda = 0, \tag{25}$$

where

$$\lambda = \frac{4 - 16\mu^2 - \omega_1^2}{9\beta^2 + \eta^2}. \tag{26}$$

The universal unfolding of the GS normal form (25) is

$$G(a, \lambda, \xi_1, \xi_2) = a^4 - \lambda + \xi_1 a + \xi_2 a^2 = 0. \tag{27}$$

It is a bifurcation of codimension two. The transition set of the universal unfolding (27) is given by:

(a) bifurcation point set, $\beta = \Phi$;

(b) hysteresis set, $H = \left\{ \left(\frac{\xi_1}{8}\right)^2 = -\left(\frac{\xi_2}{6}\right)^3 \right\}$;

(c) double limit point set, $D = \{\xi_1 = 0,\ \xi_2 \leq 0\}$;

(d) transition set, $\sum = B \cup H \cup D$.

The transition set $\sum$ divides the (ξ_1, ξ_2) plane into three regions. The topology of Eq.(27) in the unfolding parameter plane is qualitatively distinct in these various regions, as shown in Fig.1.

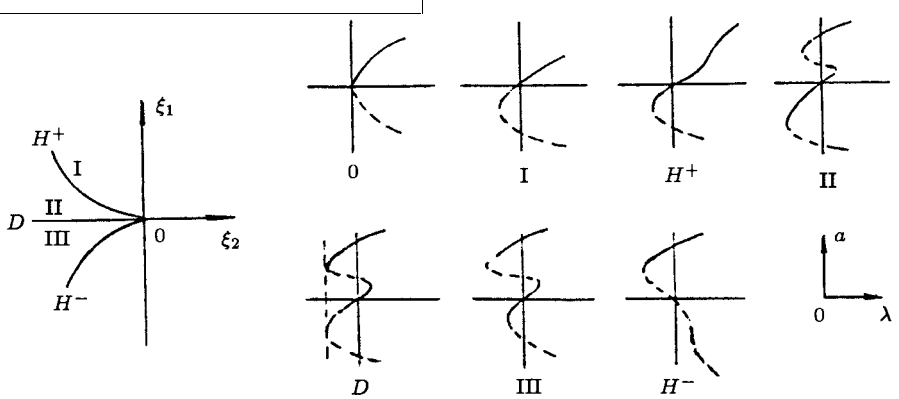


Fig.1. The transition set and bifurcation diagram.

(ii) When $4 - 16\mu^2 - \omega_1^2 = 0$ and $9\beta^2 + \eta^2 \neq 0$, the bifurcation equation (23) can be written as

$$\alpha^2 - \lambda^2 = 0 \tag{28}$$

and

$$a^2 = 0,$$

where

$$\lambda^2 = -\frac{2(4\mu\eta - 3\beta\omega_1)}{9\beta^2 + \eta^2}. \tag{29}$$

The universal unfolding of the normal form (28) is

$$G(a, \lambda, \xi) = a^2 - \lambda^2 + \xi. \tag{30}$$

It is a bifurcation of codimension one. The transition set of the universal unfolding (30) is given by:

(a) bifurcation point set, $B = \{\xi = 0\}$;

(b) hysteresis set, $H = \Phi$,

(c) double limit point set, $D = \Phi$;

(d) transition set, $\sum = B \cup H \cup D$.

The bifurcation diagram is shown in Fig.2.

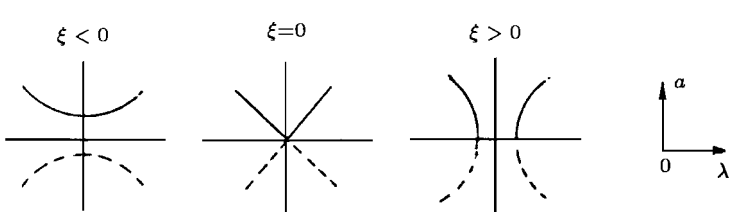


Fig.2. The bifurcation diagram.

(iii) When $9\beta^2+\eta^2 \neq 0$, the bifurcation equation (23) turns out to be

$$\tilde{a}^2-\lambda^2=0, \tag{31}$$

where

$$\tilde{a}=a^2+\frac{4\mu\eta-3\beta\omega_1}{9\beta^2+\eta^2},$$

$$\lambda=\frac{4-16\mu^2-\omega_1^2}{9\beta^2+\eta^2}+\left(\frac{4\mu\eta-3\beta\omega_1}{9\beta^2+\eta^2}\right)^2. \tag{32}$$

This case is similar to the above case.

These results of the overall analysis of the bifurcation of the parametrically excited oscillator clearly show the possibility of finding many solutions. Other bifurcation problems, for example Hopf bifurcation, are not discussed in detail.

5. Stability of the zero solution

The stability of the constant solutions is determined by the eigenvalues. We let

$$A=u+\mathrm{i}v. \tag{33}$$

Substituting Eq.(33) into Eq.(21) yields

$$\begin{aligned} u' =& -\frac{\mu}{\omega_1}u+\left(\frac{1}{4}+\frac{1}{2\omega_1}\right)v-\frac{1}{\omega_1}(\eta u+3\beta v)(u^2+v^2),\\ v' =& -\frac{\mu}{\omega_1}v+\left(-\frac{1}{4}+\frac{1}{2\omega_1}\right)u-\frac{1}{\omega_1}(\eta u-3\beta v)(u^2+v^2). \end{aligned} \tag{34}$$

The Jacobi matrix of Eq.(34) is

$$D_\omega f=\begin{bmatrix} -\dfrac{\mu}{\omega_1}-\dfrac{1}{\omega_1}(\eta\mu^2+\eta v^2+6\beta uv) & \dfrac{1}{4}+\dfrac{1}{2\omega_1}-\dfrac{1}{\omega_1}(3\beta u^2+3\beta v^2+2\eta uv)\\ -\dfrac{1}{4}+\dfrac{1}{2\omega_1}-\dfrac{1}{\omega_1}(-3\beta u^2-3\beta v^2+2\eta uv) & -\dfrac{\mu}{\omega_1}-\dfrac{1}{\omega_1}(\eta\mu^2+\eta v^2-66\beta uv)\end{bmatrix}.$$

For the zero solution, the Jacobi matrix is

$$D_\omega f=\begin{bmatrix} -\dfrac{\mu}{\omega_1} & \dfrac{1}{4}+\dfrac{1}{2\omega_1}\\ -\dfrac{1}{4}+\dfrac{1}{2\omega_1} & -\dfrac{\mu}{\omega_1}\end{bmatrix}. \tag{35}$$

The characteristic equation for the zero solution can be shown to be

$$\lambda^2+\frac{2\mu}{\omega_1}\lambda+\frac{1}{16}+\frac{\mu^2}{\omega_1^2}-\frac{1}{4\omega_1^2}=0 \tag{36}$$

The eigenvalues are

$$\lambda_{1,2}=-\frac{\mu}{\omega_1}\pm\sqrt{\frac{1}{4\omega_1^2}-\frac{1}{16}}. \tag{37}$$

(i) If $\mu\omega_1<0$, the zero solution is unstable.

(ii) If $\mu\omega_1>0$, $\dfrac{1}{\omega_1^2}-\dfrac{1}{4}>\dfrac{4\mu^2}{\omega_1^2}$, the zero solution is unstable.

6. Conclusion

In this paper we have studied a parametrically excited van der Pol–Duffing oscillator with a strong nonlinearity. The parameter transformation has been presented and the bifurcation equation determined. The analysis of the static bifurcation has given the transition set and the bifurcation diagram by using the singularity theory. These curves are not accurate numerically and are only to suggest their nature qualitatively. They completely describe the topology in various regions of the parameter plane and are associated with variations in system parameters.

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