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General explicit solutions of a classical Boussinesq system*

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Seeking a travelling wave solution of the classical Boussinesq system and making an ansatz for the solution, we obtain a nonlinear system of algebraic equations. We solve the system using an effective algorithm and then two general explicit solutions are obtained which are of physical interest.

Keywords: explicit solution, soliton, nonlinear equation
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1. Introduction

It is important to find exact solutions of nonlinear equations in nonlinear problems. Recently, a number of techniques have been proposed, such as the hyperbolic tangent function expansion,[1,2] the Jacobi elliptic function expansion,[3,4] the homogeneous balance method,[5−9] the mixing exponential method,[10] the rank analysis technique[11] and some other methods.[12−14] But there is no general method for solving a given nonlinear differential equation, especially for coupled equations.

In this paper, we consider the following classical Boussinesq system

$$\begin{cases} \eta_t + [(1+\eta)u]_x = -\dfrac{1}{4}u_{xxx}, \\ u_t + uu_x + \eta_x = 0, \end{cases} \tag{1}$$

where η is the elevation of the water wave, and u is the surface velocity of water along the x-direction. Equation (1) is one of the three sets of nonlinear long-wave equations of the Boussinesq class introduced in Ref.[15].

Applying the following transformation

$$u = -v, \qquad \eta = w - 1 - \frac{v_x}{2} \tag{2}$$

to Eq.(1) yields the Broer–Kaup (BK) system

$$\begin{cases} v_t = \dfrac{1}{2}(v^2 + 2w - v_x)_x, \\ w_t = (vw + \dfrac{1}{2}w_x)_x. \end{cases} \tag{3}$$

Reference [16] gives two basic Darboux transformations of a spectral problem associated with the BK system (3) and uses them to generate new solutions of the classical Boussinesq system (1). Here we make a simple ansatz to the solutions for Eq.(1) and we obtain the general explicit solutions which cover the solutions derived in Ref.[16].

2. General explicit solutions

To seek the travelling wave solution of Eq.(1), we make the travelling wave transformation $\eta(x,t) = \eta(\xi)$, $u(x,t) = u(\xi)$, $\xi = x - ct$ and we change Eq.(1) into the form

$$\begin{cases} -c\eta(\xi) + u(\xi)\eta(\xi) + u(\xi) + \dfrac{1}{4}u''(\xi) = 0, & \text{(4a)} \\ \eta(\xi) = cu(\xi) - \dfrac{1}{2}u^2(\xi), & \text{(4b)} \end{cases}$$

where the prime denotes $\dfrac{d}{d\xi}$. Inserting Eq.(4b) into Eq.(4a) leads to an ordinary differential equation

$$\frac{3}{2}u^2(\xi)c - \frac{1}{2}u^3(\xi) - u(\xi)c^2 + u(\xi) + \frac{1}{4}u''(\xi) = 0. \tag{5}$$

In order to solve Eq.(5), we assume that

$$u(\xi) = \frac{a_0 + a_1\tanh(k\xi) + a_2\tanh^2(k\xi)}{b_0 + b_1\tanh(k\xi) + b_2\tanh^2(k\xi)}, \tag{6}$$

where k, a_0, a_1, a_2, b_0, b_1, b_2 and $b_2 \neq 0, a_2 \neq 0$ are real constants to be determined.

Substituting Eq.(6) into Eq.(5), taking out the numerator, and setting all coefficients of $\tanh(k\xi)$ to zero yield, the following algebraic system

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$$
\begin{cases}
3\,ca_0{}^2b_1+3\,k^2a_0b_1b_2+k^2a_0b_0b_1-2\,c^2a_1b_0{}^2+4\,a_0b_0b_1+2\,a_1b_0{}^2+6\,ca_0a_1b_0 \\
\qquad -4\,c^2a_0b_0b_1-3\,a_0{}^2a_1-3\,k^2a_1b_0b_2-k^2a_1b_0{}^2=0, \\
-2\,c^2a_0b_0{}^2-k^2a_0b_2b_0-a_0{}^3+k^2a_0b_1{}^2+2\,a_0b_0{}^2-k^2a_1b_1b_0+3\,ca_0{}^2b_0+k^2a_2b_0{}^2=0, \\
k^2a_0b_2{}^2-2\,c^2a_2b_2{}^2-k^2a_1b_1b_2-k^2a_2b_0b_2-a_2{}^3+2\,a_2b_2{}^2+3\,ca_2{}^2b_2+k^2a_2b_1{}^2=0, \\
6\,ca_0a_1b_1-4\,c^2a_1b_0b_1-3\,k^2a_2b_0b_2+3\,ca_1{}^2b_0-4\,k^2a_2b_0{}^2+3\,k^2a_0b_2{}^2+2\,a_2b_0{}^2 \\
\qquad +2\,a_0b_1{}^2+3\,ca_0{}^2b_2-3\,a_0{}^2a_2-k^2a_0b_1{}^2-2\,c^2a_2b_0{}^2-4\,c^2a_0b_0b_2-3\,a_0a_1{}^2 \\
\qquad +4\,a_1b_1b_0+4\,k^2a_0b_2b_0+4\,a_0b_2b_0+k^2a_1b_1b_0-2\,c^2a_0b_1{}^2+6\,ca_0a_2b_0=0, \\
k^2a_1b_2{}^2-2\,c^2a_1b_1{}^2+6\,ca_1a_2b_0-4\,c^2a_1b_0b_2-k^2a_2b_1b_2+4\,a_0b_1b_2-a_1{}^3+6\,ca_0a_2b_1 \\
\qquad +k^2a_1b_0{}^2+4\,a_1b_0b_2+2\,a_1b_1{}^2-4\,c^2a_0b_1b_2-4\,c^2a_2b_0b_1+4\,a_2b_0b_1-k^2a_0b_0b_1 \\
\qquad -3\,k^2a_0b_1b_2+3\,ca_1{}^2b_1+6\,k^2a_1b_0b_2-3\,k^2a_2b_0b_1+6\,ca_0a_1b_2-6\,a_0a_1a_2=0, \\
3\,ca_2{}^2b_0-4\,c^2a_1b_1b_2-3\,a_0a_2{}^2-3\,a_1{}^2a_2+3\,k^2a_2b_0{}^2+4\,a_2b_0b_2-2\,c^2a_2b_1{}^2-k^2a_2b_1{}^2 \\
\qquad +6\,ca_0a_2b_2-4\,c^2a_2b_0b_2-2\,c^2a_0b_2{}^2+4\,k^2a_2b_0b_2+2\,a_2b_1{}^2+3\,ca_1{}^2b_2 \\
\qquad +k^2a_1b_1b_2+6\,ca_1a_2b_1+4\,a_1b_1b_2-3\,k^2a_0b_2b_0-4\,k^2a_0b_2{}^2+2\,a_0b_2{}^2=0, \\
-2\,c^2a_1b_2{}^2+2\,a_1b_2{}^2-3\,a_1a_2{}^2+4\,a_2b_1b_2-4\,c^2a_2b_1b_2+k^2a_2b_1b_2-3\,k^2a_1b_0b_2 \\
\qquad -k^2a_1b_2{}^2+3\,k^2a_2b_0b_1+6\,ca_1a_2b_2+3\,ca_2{}^2b_1=0.
\end{cases}
\tag{7}
$$

Now the task is to solve Eqs.(7). In order to solve Eqs.(7), one may use the Ritt–Wu elimination or the technique of Gröbner bases. However, we have instead used a much more effective algorithm. Its main idea is to consider several alternative cases, such as $a_1=0$, $a_1\neq 0$, and several subcases inside each case, etc. Applying this algorithm and carrying out all computations in the interactive mode of Maple 6, we have found the following two sets of non-trivial solutions of system (7)

$$
\begin{cases}
b_0=-\dfrac{1}{2}\,\dfrac{ca_2\pm\sqrt{-2\,b_1{}^2c^2+a_2{}^2+b_1{}^2+c^4b_1{}^2}}{-1+c^2}, \\
a_0=-a_2, \\
b_2=\dfrac{1}{2}\,\dfrac{ca_2\mp\sqrt{-2\,b_1{}^2c^2+a_2{}^2+b_1{}^2+c^4b_1{}^2}}{c^2-1}, \\
k=\sqrt{c^2-1},\ \ a_1=0
\end{cases}
\tag{8}
$$

and

$$
\begin{cases}
b_0=-\dfrac{1}{2}\,\dfrac{ca_2\pm\sqrt{-2\,b_1{}^2c^2+a_2{}^2+b_1{}^2+c^4b_1{}^2}}{-1+c^2}, \\
a_0=-a_2, \\
b_2=\dfrac{1}{2}\,\dfrac{ca_2\mp\sqrt{-2\,b_1{}^2c^2+a_2{}^2+b_1{}^2+c^4b_1{}^2}}{-1+c^2}, \\
k=-\sqrt{c^2-1},\ \ a_1=0.
\end{cases}
\tag{9}
$$

From Eqs.(8), (9) and (6), we have

$$
u(\xi)=\frac{2(c^2-1)\mathrm{sech}^2(k\xi)}{c\pm Q-2p(c^2-1)\tanh(k\xi)-(c\mp Q)\tanh^2(k\xi)} \tag{10}
$$

and

$$
u(\xi)=\frac{2(c^2-1)\mathrm{sech}^2(k\xi)}{c\pm Q+2p(c^2-1)\tanh(k\xi)-(c\mp Q)\tanh^2(k\xi)}, \tag{11}
$$

where $k=\sqrt{c^2-1}$, $Q=\sqrt{1+p^2(c^2-1)^2}$ and p, c with $c^2>1$ are arbitrary real constants.

Obviously, from Eqs.(10), (11) and (4b), we obtain the general exact solutions with two parameters for Eq.(1). Especially, taking the special value of p, some particular solutions for Eq.(1) are generated, for example:

(i) Taking $p^2=1/(c^2-1)$, we can obtain a pair of soliton solutions

$$
\begin{cases}
u(x,t)=\dfrac{(c^2-1)\mathrm{sech}^2[\sqrt{c^2-1}(x-ct)]}{c\pm\sqrt{c^2-1}\tanh[\sqrt{c^2-1}(x-ct)]}, \\
\eta(x,t)=cu(x,t)-\dfrac{1}{2}u^2(x,t),
\end{cases}
\tag{12}
$$

which have been derived by Darboux transformation in Ref.[16].

(ii) Taking $p^2 = (\alpha^2-1)/((c^2-1)(c+\alpha)^2)$, where α is an arbitrary real constant, we can obtain another pair of solutions

$$\begin{cases} u(x,t) = \dfrac{2(c^2-1)\mathrm{sech}^2[\sqrt{c^2-1}(x-ct)]}{c \pm Q - 2p(c^2-1)\tanh[\sqrt{c^2-1}(x-ct)] - (c \mp Q)\tanh^2[\sqrt{c^2-1}(x-ct)]}, \\ \eta(x,t) = cu(x,t) - \dfrac{1}{2}u^2(x,t). \end{cases} \tag{13}$$

It can be shown that Eqs.(13) are also the soliton solutions, which were derived by another new Darboux transformation in Ref.[17].

(iii) Taking $p = 0$, we can also obtain the following singular solutions which represent the shock structure

$$\begin{cases} u(x,t) = \dfrac{2(c^2-1)\mathrm{sech}^2[\sqrt{c^2-1}(x-ct)]}{(c \pm 1) - (c \mp 1)\tanh^2[\sqrt{c^2-1}(x-ct)]}, \\ \eta(x,t) = cu(x,t) - \dfrac{1}{2}u^2(x,t). \end{cases} \tag{14}$$

3. Conclusion

In summary, based on a simple ansatz we have obtained new general explicit solutions of the classical Boussinesq system (1), which can generate some soliton solutions derived by other techniques. Our approach is easy to understand and can also apply to other nonlinear evolution equations. To ensure that the result is correct, we have verified the solutions by putting them back into the original equations (1) using Maple 6.

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